Financial Mathematics Sample Exam

You are to answer these examination questions *without* consulting any notes or other resources. The exam consists of 6 problems, each worth 24 points. All sub-parts are weighted equally.

You must show your work on all problems. Partial credit will be given for all work shown, and credit will be withheld (even for correct answers) if no work is shown. No scratch paper is allowed—do all your work on the examination book and organize your results clearly.

Good luck!

1. In each case, determine whether V is a vector space. If it is not a vector space, explain why not. If it is, find basis vectors for V.

   (a) V is the subset of $\mathbb{R}^3$ defined by

   $$4x - 5y + z = 1.$$

   (b) Let the vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ represent a portfolio's holdings, where each component w_i represents the fraction of the portfolio's total market value in asset i. Let V be the set of weight vectors that can represent market-neutral long/short portfolios. The weights w_i satisfy $0 < w_i \leq 1$ for long positions, $-1 \leq w_i < 0$ for short positions, and

   $$\sum_i w_i = 0.$$

   (c) V is the set of vectors in $\mathbb{R}^2$ for which $M\mathbf{v} = \mathbf{v}$, where

   $$M = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}.$$

2. Calculate the trace and the determinant of the matrix. If the matrix is non-singular, compute its inverse. If the matrix is singular, determine its image and kernel.

   (a)
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

   (b)
$$\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$

   (c)
$$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$$

   (d)
$$M = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

   (e)
$$M = \begin{pmatrix} x & x - x^2 \\ 1 & 1 - x \end{pmatrix}$$

3. Consider the quadratic form defined by

$$Q(x, y) = 4x^2 + 24xy + 11y^2.$$

Find the maxima of $Q(x, y)$ and their location (x, y) subject to constraints as below:

(a) Max Q, subject to $x + y = 1$.

(b) Max Q, subject to $x + y = 3$.

(c) Max Q, subject to $x^2 + y^2 = 1$.

4. Suppose that a set of portfolio managers has a chance $p = 50\%$ of beating the market by 10% in a given year and chance $1 - p$ of underperforming by 10%. Performance from one year to the next is independent and uncorrelated. Returns are simple returns and compounding is ignored.

   (a) What is the probability that a manager will achieve a five-year track record which beats the market for at least 4 out of 5 years?

   (b) Suppose that unsuccessful managers get forced out of business as soon they are down overall -30%; that is, as soon as as their record contains 3 more losing years than winning years. What is the probability of failure over a five-year horizon?

   (c) Among those who survive, what is the expected total return?

5. Suppose there are 4 assets whose returns have pairwise correlations that are all equal,

$$\operatorname{Corr}(r_i, r_j) = \begin{cases} 1, & i = j, \\ \rho, & i \neq j. \end{cases}$$

Then the correlation matrix is given by

$$C = \begin{pmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{pmatrix}$$

with eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

(a) What are the eigenvalues of C?

(b) What is the trace of C?

(c) What is the determinant of C?

(d) What are the allowed values of ρ for C to be a valid correlation matrix? (*Hint: recall that a correlation matrix is always positive semi-definite.*)

6. Let X be a random variable with a uniform distribution on the finite interval $[-1, \theta]$, where $\theta > -1$ is unknown. Suppose that a random sample of size n is drawn from the distribution, with observations $x_1, \ldots, x_n$.

   Write down the likelihood function for the parameter θ, and find the maximum likelihood estimate (MLE) for θ.