Financial Mathematics Sample Exam – Solutions

You are to answer these examination questions *without* consulting any notes or other resources. The exam consists of 6 problems, each worth 24 points. All sub-parts are weighted equally.

You must show your work on all problems. Partial credit will be given for all work shown, and credit will be withheld (even for correct answers) if no work is shown. No scratch paper is allowed—do all your work on the examination book and organize your results clearly.

Good luck!

1. In each case, determine whether V is a vector space. If it is not a vector space, explain why not. If it is, find basis vectors for V.

   (a) V is the subset of $\mathbb{R}^3$ defined by

   $$4x - 5y + z = 1.$$

   (b) Let the vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ represent a portfolio's holdings, where each component w_i represents the fraction of the portfolio's total market value in asset i. Let V be the set of weight vectors that can represent market-neutral long/short portfolios. The weights w_i satisfy $0 < w_i \leq 1$ for long positions, $-1 \leq w_i < 0$ for short positions, and

   $$\sum_i w_i = 0.$$

   (c) V is the set of vectors in $\mathbb{R}^2$ for which $M\mathbf{v} = \mathbf{v}$, where

   $$M = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}.$$

   Solution:

   (a) V is not a vector space since it does not contain the origin (i.e., the zero vector).

   (b) V is not a vector space because it is not closed under scalar multiplication or addition, which violate the inequality as well as the budget constraint.

   (c) M has eigenvalues of 1 and 2, so the eigenvector

   $$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

   is a basis for the vector space V.

2. Calculate the trace and the determinant of the matrix. If the matrix is non-singular, compute its inverse. If the matrix is singular, determine its image and kernel.

   (a)
   $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

   (b)
   $$\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$

   (c)
   $$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$$

   (d)
   $$M = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

   (e)
   $$M = \begin{pmatrix} x & x - x^2 \\ 1 & 1 - x \end{pmatrix}$$

   Solution:

   (a)
   $$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \ \mathrm{Tr}\, M = 5,\ \mathrm{Det}\, M = -2,\ M^{-1} = \frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$$

   (b)
   $$M = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}, \ \mathrm{Tr}\, M = 11,\ \mathrm{Det}\, M = 0,\ \mathrm{Im}\, M = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}, \ \mathrm{Ker}\, M = \mathrm{span}\left\{\begin{pmatrix} 4 \\ -3 \end{pmatrix}\right\}$$

   (c)
   $$M = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}, \ \mathrm{Tr}\, M = 2,\ \mathrm{Det}\, M = 1,\ M^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

   (d) M is non-singular provided $\rho^2 \neq 1$.
   $$M = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \ \mathrm{Tr}\, M = 2,\ \mathrm{Det}\, M = 1 - \rho^2,\ M^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$$

   (e)
   $$M = \begin{pmatrix} x & x - x^2 \\ 1 & 1 - x \end{pmatrix}, \ \mathrm{Tr}\, M = 1,\ \mathrm{Det}\, M = 0,\ \mathrm{Im}\, M = \mathrm{span}\left\{\begin{pmatrix} x \\ 1 \end{pmatrix}\right\}, \ \mathrm{Ker}\, M = \mathrm{span}\left\{\begin{pmatrix} 1-x \\ -1 \end{pmatrix}\right\}$$

3. Consider the quadratic form defined by

$$Q(x, y) = 4x^2 + 24xy + 11y^2.$$

Find the maxima of $Q(x, y)$ and their location (x, y) subject to constraints as below:

(a) Max Q, subject to $x + y = 1$.
(b) Max Q, subject to $x + y = 3$.
(c) Max Q, subject to $x^2 + y^2 = 1$.

Solution:

(a) The Lagrange function is

$$L = Q(x, y) + \lambda(x + y - 1),$$

so taking partial derivatives gives three linear equations in three unknowns,

$$\frac{\partial L}{\partial x} = 8x + 24y + \lambda = 0,$$
$$\frac{\partial L}{\partial y} = 24x + 22y + \lambda = 0,$$
$$\frac{\partial L}{\partial \lambda} = x + y - 1 = 0.$$

The solution is

$$x = 1/9, \quad y = 8/9, \quad \lambda = -200/9, \quad Q_{max} = 100/9$$

(b) The only change is the constant of the constraint equation,

$$\frac{\partial L}{\partial \lambda} = x + y - 3 = 0.$$

The solution is

$$x = 1/3, \quad y = 8/3, \quad \lambda = -200/3, \quad Q_{max} = 100$$

Note that if we define $x = 3x'$ and $y = 3y'$, then the constraint is $x' + y' = 1$ as above, while the quadratic form scales as $Q(x, y) = 9Q(x', y')$, so that Q_{max} is 9 times larger than above.

(c) With a quadratic constraint, the equations change slightly but become more complex

$$\frac{\partial L}{\partial x} = 8x + 24y + 2\lambda x = 0,$$
$$\frac{\partial L}{\partial y} = 24x + 22y + 2\lambda y = 0,$$
$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0.$$

Multiply the first equation by y and the second by x and then subtract to eliminate λ, leaving

$$\begin{aligned} 12x^2 + 7xy - 12y^2 = 0 = (3x + 4y)(4x - 3y), \\ x^2 + y^2 = 1. \end{aligned}$$

So the solution is

$$(x, y, \lambda) = \pm(3/5, 4/5, -20), \quad Q_{max} = 20.$$

4. Suppose that a set of portfolio managers has a chance $p = 50\%$ of beating the market by 10% in a given year and chance $1 - p$ of underperforming by 10%. Performance from one year to the next is independent and uncorrelated. Returns are simple returns and compounding is ignored.

   (a) What is the probability that a manager will achieve a five-year track record which beats the market for at least 4 out of 5 years?

   (b) Suppose that unsuccessful managers get forced out of business as soon they are down overall -30%; that is, as soon as as their record contains 3 more losing years than winning years. What is the probability of failure over a five-year horizon?

   (c) Among those who survive, what is the expected total return?

   Solution:

   (a) The chance of beating the market at least $4/5$ years is $18.75\% = 6/32$. The chance of 5 winners in a row is $1/32$; the chance of 4 out of 5 is equal to $5/32$ since there are 5 different years in which the a single loss could occur; and since these are mutually exclusive, their probabilities add.

   (b) The probability of "ruin" is $7/32 = 22\%$. This is an example of the "absorbing barrier." The track records, or sample paths, that lead to going out of business can be grouped by the number of initial losses that precede the first success. They are

   - $FFFxx$ (4/32)
   - $FFSFF$ (1/32)
   - $FSFFF$ (1/32)
   - $SFFFF$ (1/32)

   (c) Among the survivors, the expected return will be positive, an effect known as "survivorship bias," which is just a consequence of conditional probability. The overall chance and returns are

   - 5 winning years (1/32, +50%)
   - 4 winning years (5/32, +30%)
   - 3 winning years (10/32, +10%)
   - 2 winning years (9/32, -10%, since one of the $\binom{5}{2}$ patterns, $FFFSS$, is absent because it is not a survivor having lost the first 3 years in a row)

   Applying the rules of conditional probability, the desired expectation is

$$\mu = \frac{(1/32)(0.5) + (5/32)(0.3) + (10/32)(0.1) + (9/32)(-0.1)}{(1/32) + (5/32) + (10/32) + (9/32)} = 8.4\%$$

5. Suppose there are 4 assets whose returns have pairwise correlations that are all equal,

$$\operatorname{Corr}(r_i, r_j) = \begin{cases} 1, & i = j, \\ \rho, & i \neq j. \end{cases}$$

Then the correlation matrix is given by

$$C = \begin{pmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{pmatrix}$$

with eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

(a) What are the eigenvalues of C?

(b) What is the trace of C?

(c) What is the determinant of C?

(d) What are the allowed values of ρ for C to be a valid correlation matrix? (*Hint: recall that a correlation matrix is always positive semi-definite.*)

Solution:

(a) The eigenvalues are $\lambda_1 = 1 + 3\rho$ and $\lambda_j = 1 - \rho$ for $j = 2, 3, 4$.

(b) $\operatorname{Tr} C = 4$.

(c) $\operatorname{Det} C = \prod \lambda_i = (1 + 3\rho)(1 - \rho)^3$.

(d) Since C is a correlation matrix, the eigenvalues must all be non-negative. So in addition to the usual restriction $-1 \leq \rho \leq 1$, we have $\lambda_1 \geq 0$. Therefore

$$-\frac{1}{3} \leq \rho \leq 1.$$

6. Let X be a random variable with a uniform distribution on the finite interval $[-1, \theta]$, where $\theta > -1$ is unknown. Suppose that a random sample of size n is drawn from the distribution, with observations $x_1, \ldots, x_n$.

   Write down the likelihood function for the parameter θ, and find the maximum likelihood estimate (MLE) for θ.

   Solution: The likelihood function is

$$\mathcal{L} = \frac{1}{(\theta + 1)^n}.$$

   The likelihood is maximized when $(\theta + 1)$ is minimized. The possible estimates are constrained by the data, which respects

$$-1 \leq \min_i\{x_i\} \leq \max\{x_j\} \leq \theta.$$

   So the MLE is

$$\theta = \max_j\{x_j\}.$$