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OPTIMAL INVESTMENT IN PRODUCT-FLEXIBLE MANUFACTURING CAPACITY*

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This paper presents a model and an analysis of the cost-flexibility tradeoffs involved in investing in product-flexible manufacturing capacity. Flexible capacity provides a firm with the ability to respond to a wide variety of future demand outcomes, but at the expense of the increased cost of acquiring flexible manufacturing capacity, as compared with dedicated or nonflexible capacity. We formulate the product-flexible manufacturing capacity investment decision as a two-stage stochastic program. In the first stage, the firm must make its investment decision in manufacturing capacity, before the resolution of uncertainty in product demand. In the second stage, after demand for products are known, the firm implements its production decisions, constrained by the first-stage investments.

The main contributions of this paper are threefold. First, we develop a model of the firm's flexible manufacturing investment decision that conceptually captures some of the key characteristics of this complex decision problem. Second, with the aid of the model, we characterize the necessary and sufficient conditions for a firm to invest in flexible capacity to protect efficiently against uncertainty in demand for all of its products. Third, we explore the sensitivity of the firm's optimal capacity investment decision to key problem components, namely to the cost of flexible and nonflexible production capacity, to the underlying distribution of product demand, and to the level of risk.
(TECHNOLOGY INVESTMENT; FLEXIBLE MANUFACTURING SYSTEMS)

1. Introduction and Literature Review

Many firms are finding that their available tools for considering cost/benefit tradeoffs for investments in flexible automation often contradict the intuition of their managers, many of whom perceive significant benefits from acquiring flexible manufacturing systems (FMS's). Kaplan (1986) acknowledges that even a very careful application of discounted cash flow techniques to evaluate a potential investment in flexible automation will not capture the strategic benefits of flexibility. Kaplan proposes that managerial judgement be applied to decide whether the strategic benefits of an investment in flexible automation outweigh the difference, or gap, between the investment cost and the quantifiable benefits.

In this paper we attempt to narrow this technology evaluation gap, in which managers have insufficient tools to support their investment decisions. We develop a model that focuses on the economic tradeoffs between the acquisition costs of flexible capacity and a firm's ability to respond flexibly to future uncertain demand. We then use this model to characterize the necessary and sufficient conditions for the acquisition of flexible capacity. We also use the model to examine the sensitivity of the firm's optimal capacity investment decision to the costs of capacity, to the distribution of demand, and to the level of risk.

We focus our analysis on the use of flexible capacity to hedge against uncertainty in future demand. Other motives for investing in flexible technology include developing the ability to rapidly introduce new product models, reducing the need for interperiod inventories, and expanding product scope to invade competitors' markets. (Incorporating all of these factors into one model would be quite difficult at this time.) We view our

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analysis as contributing to the theory necessary to develop realistic models for complete analysis of FMS investment decisions.

The subject of the economics of flexibility has been of interest to economists for a long time (see e.g., Stigler 1939, and the many references in Jones and Ostroy 1984), but has become a significant interest in the management science community only recently, following the increasing viability of flexible, computer-controlled manufacturing systems. This new interest has spurred a significant amount of work in a very short time. (See, e.g., Stecke and Suri 1984, 1986, Adler 1985, Buzacott and Yao 1986, Fine 1990.)

Some recent work (Pindyck 1988, Majd and Pindyck 1987, Jones and Ostroy 1984) explores flexibility only as it relates to the timing structure of capacity acquisition, information acquisition, and commitment of resources: A firm loses flexibility when it makes an irreversible commitment. These papers do not deal with technologies that exhibit flexibility per se. The work of Gaimon (1986, 1988) and Roth, Gaimon, and Krajewski (1986) attempts to move in this direction. These model formulations do not explicitly capture a technology's flexibility characteristics; however, they attribute characteristics often associated with the acquisition of flexible technology (e.g., greater revenue resulting from broader product mix capabilities and lower operating costs due to economies of scope) to the flexible automation that is acquired.

Our work, as well as Burstein (1986), Kulatilaka (1988), Fine and Li (1988), Fine and Pappu (1988), Caulkins and Fine (1988), and Graves (1988), considers explicitly the use of a flexible production technology. Burstein presents a deterministic, dynamic, two-product, mixed integer programming model with flexible automation that is subject to technological improvement over time. He characterizes optimal technology acquisition policies. Kulatilaka's model is a stochastic, dynamic program that analyzes optimal use of a flexible technology that can be utilized and switched among a variety of operating modes, depending on the outcome of demand in each period. Fine and Li explore how the dynamics of the life cycles of multiple products affect incentives for investing in flexible capacity that exhibits economies of scope in investment costs. Fine and Pappu use a game-theoretic model to show how the existence of flexibility can intensify competition among firms. Caulkins and Fine analyze flexible capacity as a substitute or complement to seasonal inventories. Graves analyzes the interaction between safety stocks and flexible manufacturing capability.

Another recent strand of the flexibility literature (Karmarkar and Kekre 1987; Vander Veen and Jordan 1987) explores the relative advantages of buying many small machines for dedicated use versus buying fewer large machines to be used flexibly. The analysis focuses on the tradeoff between the differences in investment costs for the different options and the inventory and changeover costs (in added capacity needs) required for the latter option.

Finally, the option pricing models from the finance literature (see, e.g., the survey by Mason and Merton 1985) can be used for analyzing investments under uncertainty. Although these models have been available for over ten years, and are used extensively in the analysis of investments in financial securities, they do not seem to have taken hold in the capital budgeting procedures of most corporations. One reason for this may be the perceived complexity of the model underlying the option pricing formula. (However, the work of Kulatilaka 1985, 1988 may speed the adoption of these techniques for evaluation of flexible automation.)

§2 presents our model formulation and observations on its solution. In §3 we present results on the necessary and sufficient conditions for purchasing flexible capacity, namely: Flexible capacity should be acquired when the expected value of its best usage in each possible future state, summed over all states, exceeds its costs. §4 examines the sensitivity of the optimal net revenue and the optimal capacity investment levels for flexible and nonflexible technologies to the costs of various capacity purchase options. Our results indicate that as a function of capacity acquisition costs, optimal net revenues are convex and nonincreasing, and optimal capacity levels are monotone. We also show some subtle but important cross-elasticity results regarding changes in optimal capacity levels to changes in capacity costs. §5 explores (via a numerical example) the sensitivity of optimal investment in FMS capacity to negatively and positively correlated demand, and to the level of risk in the demand distribution. Our results indicate that negatively (positively) correlated product demands enhance (detract from) the value of flexible capacity and that the level of demand riskiness does not always affect monotonically the optimal quantity of flexible capacity. We provide concluding remarks in §6.

2. Model Formulation

We formulate our product-flexible capacity investment model as a two-stage stochastic program. In the first stage the firm makes its technology investment decisions. It then observes a random variable that affects demand, and in the second stage, makes its production decisions. The model structure enables us to focus on an important characteristic of flexible manufacturing technologies: The lead time for adjusting the usage of a flexible technology is much shorter than the lead time for investing in and installing new or upgraded facilities. The underlying simplification employed here is that we only consider one period of production after the technology acquisition stage. The relaxation of this simplifying assumption is considered by Caulkins and Fine (1989), who develop a dynamic production model with interperiod inventories. In essence, the model presented here rolls all future production and inventory decisions into one future period, as is also done, for example, by Stigler (1939) and Jones and Ostroy (1984).

We model the capacity investment problem as a single firm optimization problem. We assume that the firm can sell n different product families, indexed by $j = 1, \ldots, n$. The firm has $n + 1$ types of capacity available to it: dedicated j capacity, for $j = 1, \ldots, n$, each of which can manufacture only members of its own product family, and flexible capacity (indexed by F) that can produce any or all of the n product families. (In essence, dedicated (or nonflexible) capacity incurs an infinite cost of product changeover whereas flexible capacity incurs zero changeover cost.)

We denote by K_j for $j = 1, \ldots, n$ and K_F the amounts of dedicated and flexible capacity purchased by the firm. Let $r_j(K_j)$, $j = 1, \ldots, n$ and $r_F(K_F)$ denote the acquisition costs for these technologies. In practice, $r_j(\cdot)$ and/or $r_F(\cdot)$ may typically be concave or linear, or even discontinuous. For each j, we assume $r_F(K) > r_j(K) > 0$ for $K > 0$. That is, acquiring K units flexible capacity costs more than acquiring K units of any one type of dedicated capacity.

After making its capacity investment decisions, the firms learns information about the state of the world for market demand for its product families. There are k possible states of the world; state i occurs with probability p_i ($p_i > 0$ for all i and $\Sigma_i p_i = 1$). For each realization i, the firm chooses production levels for each product family, subject to the capacity constraints imposed by the investment decisions. We denote by Y_{ij} the total quantity of product j produced on the dedicated j capacity if state i occurs, and by Z_{ij} the total quantity of product j produced in state i on the flexible capacity.

Given that state i is realized, the firm's revenue $R_{ij}(\cdot)$ from product j is a function of $X_{ij} \equiv Y_{ij} + Z_{ij}$, the total quantity of product j to be manufactured and sold. We use primes to denote derivatives of $R_{ij}(\cdot)$.

The firm incurs a cost $C_j(Y_{ij})$ for producing Y_{ij} units of product j using dedicated j capacity and $C_{jF}(Z_{ij})$ for producing Z_{ij} units of product j using flexible capacity. The cost functions $C_j(\cdot)$ and $C_{jF}(\cdot)$ in practice may be convex, linear, concave, or neither.

The following then is a model of the product-flexible manufacturing system (PFMS) investment problem:

$$\text{maximize} \quad -r_F(K_F) - \sum_j r_j(K_j), \quad +\sum_i p_i \sum_i \{R_{ij}(Y_{ij} + Z_{ij}) - C_j(Y_{ij}) - C_{jF}(Z_{ij})\},$$

$$K_F, K_j; \quad j = 1, \ldots, n, \qquad Y_{ij}; Z_{ij}; \quad i = 1, \ldots, k, j = 1, \ldots, n$$

$$\text{subject to:} \quad Y_{ij} - K_j \le 0, \quad j = 1, \ldots, n; \quad i = 1, \ldots, k,$$

$$\sum_j Z_{ij} - K_F \le 0, \quad i = 1, \ldots, k,$$

$$K_F \ge 0, \quad K_j \ge 0, \quad Y_{ij} \ge 0, \quad Z_{ij} \ge 0; \quad j = 1, \ldots, n; \quad i = 1, \ldots, k.$$

We assume that for all i and j, that the revenue function $R_{ij}(\cdot)$ is differentiable, strictly concave, and has a unique, finite, positive maximand, Q^*_{ij}. (These assumptions all hold, for example, when the firm faces a downward-sloping linear demand curve.) If we also assume that the capacity acquisition cost functions $r_F(\cdot)$ and $r_j(\cdot)$ and the production cost functions $C_j(\cdot)$ and $C_{jF}(\cdot)$ are increasing and continuous, then it is straightforward to show that this problem has an optimal solution. However, because the objective function of the model is not necessarily convex there may be more than one optimal solution, there may be isolated optimal solutions, and the set of optimal solutions may be discontinuous and otherwise badly behaved in the problem parameters. In the interests of gaining insight into the PFMS investment problem, we therefore will need to make a number of simplifying assumptions about the model that will make the model more amenable to analysis.

We therefore assume that capacity acquisition costs are linear, i.e., $r_F(K_F) = r_F K_F$ and $r_j(K_j) = r_j K_j$, and that $r_F > r_j > 0, j = 1, \ldots, n$. That is, flexible capacity costs more per unit than any one dedicated capacity. We further assume that all variable production costs are linear and are technology independent, i.e., $C_{jF}(x) = C_j(x) = C_j \cdot x, j = 1, \ldots, n$. For technologies where most of the variable costs are materials costs, this assumption is reasonable. For example, when the investment problem is to compare the alternatives of dedicated, highly-automated capacity with flexible, highly-automated capacity, the labor content for either technology is very low, so the material costs will dominate and the variable costs will be very similar for the two technologies. The principal cost difference between the two technologies lies in the up-front investment costs, captured by $r_j, j = 1, \ldots, n$ and r_F.

These assumptions yield the following PFMS model formulation:

$$\text{maximize} \quad -r_F K_F - \sum_j r_j K_j \quad +\sum_i p_i \sum_j \{R_{ij}(Y_{ij} + Z_{ij}) - C_j \cdot (Y_{ij} + Z_{ij})\},$$

$$\{K_F, K_j; j = 1, \ldots, n\}$$

$$\{Y_{ij}; Z_{ij}; i = 1, \ldots, k, j = 1, \ldots, n\}$$

$$\begin{aligned}
\text{subject to} \quad & Y_{ij} - K_j \le 0, & i = 1, \ldots, k \quad & j = 1, \ldots, n, & (\lambda_{ij}) \\
& \sum_j Z_{ij} - K_F \le 0, & i = 1, \ldots, k, \quad & & (\gamma_i) \\
& Y_{ij} \ge 0, & i = 1, \ldots, k, \quad & j = 1, \ldots, n, & (s_{ij}) \\
& Z_{ij} \ge 0, & i = 1, \ldots, k \quad & j = 1, \ldots, n, & (t_{ij}) \\
& K_j \ge 0, & j = 1, \ldots, n, \quad & & (u_j) \\
& K_F \ge 0. & & & (v)
\end{aligned}$$

(The Karush-Kuhn-Tucker (K-K-T) multipliers appear to the right of each constraint.)

It is straightforward to show that there is an optimal solution to this model.

3. Necessary and Sufficient Conditions for Purchasing Flexible Capacity

Let $\bar{K} \equiv (\bar{K}_1, \ldots, \bar{K}_n, \bar{K}_F)$ and $(\bar{Y}, \bar{Z}) \equiv (\bar{Y}_{ij}, \bar{Z}_{ij}; i = 1, \ldots, k, j = 1, \ldots, n)$ denote an optimal solution to the above model. The strict concavity of the revenue functions assures that any optimal solution is unique in the quantities $\bar{X}_{ij} \equiv \bar{Y}_{ij} + \bar{Z}_{ij}$. (With more than two products, there is no guarantee of uniqueness of the optimal solution in the capacity decisions. The next section and the Appendix further discuss this issue.) The nonuniqueness in $\bar{Y}_{ij}$ and $\bar{Z}_{ij}$ separately can arise, for example, if there is excess flexible capacity ($\sum_j \bar{Z}_{ij} < \bar{K}_F$) and excess dedicated j capacity ($\bar{Y}_{ij} < \bar{K}_j$) in state i. In such a case $\bar{X}_{ij}$, the optimal quantity of product j in state i, can be produced in more than one way. We first observe:

PROPOSITION 1 (*Existence and Uniqueness of Solutions to the PFMS Problem*).
1. *The model PFMS has a solution* $\bar{K}, \bar{Y}, \bar{Z}$.
2. *The production quantities* $\bar{X}_{ij} = \bar{Z}_{ij} + \bar{Y}_{ij}$ *are unique.*
3. *If there are only two products* ($n = 2$), *the optimal investment decision* $\bar{K}$ *is unique.*
4. *For three or more products* ($n \geq 3$) *there is no guarantee that* $\bar{K}$ *is unique.*

PROOF. See the Appendix.

LEMMA 1 (*Shadow Values at Optimality*). *Let* $\bar{K}, \bar{Y}, \bar{Z}$ *be an optimal solution to the PFMS model. The following multipliers are optimal shadow values on the capacity constraints*:

(a) $\bar{\lambda}_{ij} = p_i[R'_{ij}(\bar{Y}_{ij} + \bar{Z}_{ij}) - C_j]^+, i = 1, \ldots, k; j = 1, \ldots, n,$
(b) $\bar{\gamma}_i = \max_j(\bar{\lambda}_{ij}), i = 1, \ldots, k.$

PROOF. See the Appendix.

The quantity $\bar{\lambda}_{ij}$, the shadow value of the dedicated capacity constraint, is the marginal value of production of product j in state i. Part (b) of Lemma 1 states that the shadow value of flexible capacity in state i is equal to the maximum, over all products j, of the shadow values of the dedicated capacities in state i. Because the flexible capacity can be used to produce any of the n products, it will be used, in each state i, for the product that yields the highest marginal profit. This intuition leads us to our next result, on necessary and sufficient conditions for purchasing a positive amount of flexible capacity.

Before turning to that result, we must first consider the PFMS problem when K_F is unavailable or has an extremely large cost, i.e., $r_F >> r_j$, for all j. In the absence of flexible capacity, the firm faces n of the following independent decision problems of how much j-capacity to purchase:

$$P(j): \quad \underset{K_j, Y_{ij}}{\text{maximize}} \quad -r_j K_j + \sum_{i=1}^{k} p_i[R_{ij}(Y_{ij}) - C_j \cdot Y_{ij}]$$

$$\text{subject to:} \quad Y_{ij} - K_j \leq 0 \quad (\lambda_{ij}),$$

$$Y_{ij} \geq 0 \quad (s_{ij}),$$

$$K_j \geq 0 \quad (v_j).$$

These subproblems $P(j)$, for $j = 1, \ldots, n$, are amenable to easy solution. In fact, when the revenue functions are quadratic (i.e., demand curves are linear), the problems $P(j)$ have closed-form solutions. (See Freund and Fine 1986.)

THEOREM 1 (*Necessary and Sufficient Conditions for Purchasing FMS Capacity*). *Let the independent subproblems $P(j)$ be solved with solutions $\bar{K}_j$, $\bar{Y}_j$, and let λ^*_{ij}, s^*_{ij}, v^*_j be the optimal K-K-T multipliers for the subproblems $P(j)$, $i = 1, \ldots, k, j = 1, \ldots, n$. Then, for the PFMS problem,*

(a) *$\bar{K}_F > 0$ whenever $\sum_i \max_j (\lambda^*_{ij}) > r_F$*

(b) *$\bar{K} = 0$ is optimal whenever $\sum_i \max_j (\lambda^*_{ij}) < r_F$.*

(*Note. At equality, i.e., $\sum_i \max_j (\lambda^*_{ij}) = r_F$, $\bar{K}_F > 0$ and $\bar{K}_F = 0$ can each be optimal, because we cannot guarantee a unique solution. However, with only two products the value of K is unique (Proposition 1, part 3), so the condition $\sum_i \max_j (\lambda^*_{ij}) > r_F$ is necessary and sufficient for $\bar{K}_F > 0$.*)

PROOF. See the Appendix.

Theorem 1 is of interest because it permits one to test at what cost flexible capacity becomes economical by analyzing only the easy-to-solve subproblems $P(j)$. Furthermore, the results (a) and (b) are quite intuitive: Flexible capacity should be acquired when the expected value of its best usage in each state, summed over the states, exceeds its cost.

This insight is not particularly surprising given the model formulation. However, it is the model formulation, in our opinion, that makes this intuition easy to understand, and consequently of potential use to managers focusing on the flexible capacity acquisition decision.

This potential usefulness is illustrated by the following example: We are aware of a major U.S. corporation that, as part of its capacity planning process, identifies possible future states of the world, assesses the likelihood of each state, and then sets its capacities and technology plans under the assumption that the most likely state will occur. If one accepts the objective of maximizing expected profits, our model provides a clearly superior approach: the prescribed capacity and technology acquisition decision optimizes net expected benefits by explicitly hedging against the different possible states of the world. Insurance against events other than the most likely one is achieved at minimum cost.

4. Properties of the Optimal Value Function and the Optimal Capacity Levels

In this section we explore the sensitivity of the optimal value function and the optimal capacity levels to changes in the per unit costs of capacity. We first examine the sensitivity of the optimal value function of the PFMS problem to the capacity costs, r_j, for $j = 1, \ldots, n$ and r_F. Let $r = (r_1, \ldots, r_n, r_F) \in R^{n+1}_+$ and let $z^*(r)$ be the optimal value function of the PFMS problem for a given vector r of unit capacity costs. The following Lemma characterizes basic properties of the optimal value function $z^*(r)$.

LEMMA 2 (*Characterization of the Optimal Value Function*). *For the PFMS problem,*

1. *$z^*(r)$ is convex in r.*
2. *$z^*(r)$ is nonincreasing in r_j for $j = 1, \ldots, n$, and is nonincreasing in r_F.*
3. *If $\bar{K}_j$ is unique at r, then $\partial z^*(r)/\partial r_j = -\bar{K}_j, j = 1, \ldots, n$. If $\bar{K}_F$ is unique at r, then $\partial z^*(r)/\partial r_F = -\bar{K}_F$.*
4. *If $\bar{K} = (\bar{K}_1, \ldots, \bar{K}_n, \bar{K}_F)$ is not unique at r_j then $-\bar{K} = (-\bar{K}_1, \ldots, -\bar{K}_n, -\bar{K}_F)$ is a subgradient of $z^*(r)$ at r.*

PROOF. See the Appendix.

(Note that part 3 of Lemma 2 is a slight generalization of the envelope theorem (see, e.g., Varian 1978) to the case in which $z^*(r)$ is not necessarily differentiable.)

The result that $z^*(r)$ is nonincreasing in r is quite intuitive; decreased costs of capacity acquisition increase the maximum profit the firm can earn. The convexity of $z^*(r)$ suggests that as the cost of acquiring advanced manufacturing technology continues its decline, firms will enjoy increasingly large profit improvements. Part 3 of the Lemma tells us exactly how rapidly the profit improves as capacity acquisition costs decline. Such a result is clearly useful for evaluating potential returns to reductions in the acquisition costs.

Our next result shows the sensitivity of the optimal capacity decisions to changes in the capacity costs. As was shown in Proposition 1 of the previous section, the uniqueness of the optimal capacity levels can only be guaranteed when the model is restricted to two product families ($n = 2$). Therefore our results concerning the sensitivity of the optimal capacity decisions are of necessity limited to the case of two product families. Nevertheless, there are a number of settings in which the two-product families capture quite well the essence of the problem application.

The automobile industry provides two examples: Automobile companies planning their plant capacity for the manufacture of auto chassis and transmissions have significant concern over the future demand mix of front-wheel-drive and rear-wheel-drive vehicles. The natural interpretation of our model for this problem is to determine the economic viability of building plants flexible enough to produce both front-wheel-drive and rear-wheel-drive chassis and transmissions in light of the demand mix uncertainty, relative to the option of building only dedicated front-wheel-drive and/or rear-wheel-drive plants.

A similar application, mentioned by Gold (1982) for Ford Motor Company's automobile engine plants, arises in the production of six and eight cylinder engines. In this case, uncertainty about future oil prices (which influence the demand for fuel efficiency in automobiles) makes holding flexible capacity a potentially valuable option.

Finally, with slight modification, our model formulation can handle uncertainty about the supply of factors of production rather than (or in addition to) uncertainty about demand. (This extension is easily accomplished by making the variable production costs depend on the states.) For this modification, the electric power industry (which faces uncertainty concerning the relative prices of oil and coal) could potentially employ the model with the restriction to two energy supply options.

The next result shows the sensitivity of the optimal capacity decisions to changes in the capacity costs for two-product-family case. For ease of exposition, we hereafter denote the two product families by A and B, with respective capacity acquisition costs r_A and r_B. In addition, we denote the flexible capacity quantity K_F by K_{AB} and the per unit cost of flexible capacity r_F as r_{AB}.

THEOREM 2 (*Sensitivity of the optimal capacity levels to the acquisition costs*). *For the two-product PFMS problem, if* $\bar{K}(r) > 0$, *then*

(1) (A) $\bar{K}_A$ *is strictly decreasing in* r_A,
(B) $\bar{K}_B$ *is strictly decreasing in* r_B,
(AB) $\bar{K}_{AB}$ *is strictly decreasing in* r_{AB},
(2) (AB) $\bar{K}_A$ *is strictly increasing in* r_{AB}, $\bar{K}_B$ *is strictly increasing in* r_{AB},
(A) $\bar{K}_{AB}$ *is strictly increasing in* r_A,
(B) $\bar{K}_{AB}$ *is strictly increasing in* r_B,
(3) (A) $\bar{K}_B$ *is strictly decreasing in* r_A,
(B) $\bar{K}_A$ *is strictly decreasing in* r_B,
(4) (A) $\bar{K}_A + \bar{K}_{AB}$ *is decreasing in* r_{AB},
(B) $\bar{K}_B + \bar{K}_{AB}$ *is decreasing in* r_{AB},
(5) (A) $\bar{K}_A + \bar{K}_{AB}$ *is decreasing in* r_A,
(B) $\bar{K}_{AB} + \bar{K}_B$ *is increasing in* r_A,
(6) (A) $\bar{K}_B + \bar{K}_{AB}$ *is decreasing in* r_B, *and*
(B) $\bar{K}_{AB} + \bar{K}_A$ *is increasing in* r_B.

PROOF. See the Appendix.

Statements (1) and (2) of the theorem are quite intuitive. As r_{AB}, the cost of flexible capacity, decreases (relative to r_A and r_B), the firm substitutes flexible capacity for the two types of dedicated capacity ((1)(AB) and (2)(AB)). An increase in r_A, the cost of dedicated A capacity, leads the firm to decrease the amount of A capacity acquired ((1)(A)), and replace it with flexible capacity ((2)(A)). Statement (3)(A) provides further insight: Because the increase in r_A leads to an increase in flexible capacity, the firm will also decrease its dedicated B capacity because the flexible capacity substitutes for it as well as for the A capacity. (Of course, symmetric results hold when we switch the roles of A and B in this discussion.) Statement (3) is a useful result for sharpening one's intuition about the model formulation and the flexible capacity decision problem. Most readers will find this "ripple substitution" effect sensible and intuitive once it is presented to them. However, it is our experience that, without the aid of the model and its analysis, many people do not have a well-developed intuition with respect to this phenomenon.

Whereas statements (1), (2), and (3) of the theorem provide directional properties of the optimal capacity levels, statements (4), (5), and (6) sharpen the results by providing a ripple result on relative magnitudes of these effects. For example, statements (1)(AB), (2)(AB), and (4)(A) together imply that, with an increase in r_{AB}, the magnitude of the decrease in $\bar{K}_{AB}$ is greater than the magnitude of the increase in $\bar{K}_A$. Furthermore, statements (1)(A), (2)(A), (3)(A), and (5) imply that with an increase in r_A, the magnitude of the decrease in $\bar{K}_A$ exceeds the magnitude of the increase in $\bar{K}_{AB}$, which in turn, exceeds the magnitude of the decrease in $\bar{K}_B$. Thus for a perturbation in r_A, the ripples decrease in size from dedicated A capacity to flexible capacity to dedicated B capacity.

Although not immediately generalizable to the n product case, these results aid our understanding of how a change in the cost parameter for one type of technology can have implications for a firm's entire optimal technology portfolio. The existence of a flexible technology serves to integrate and render more complex an investment problem that otherwise would break into independent, easily-solvable components. If the flexible technology did not exist, a change in r_A would not affect the optimal K_B. In compensation for the added complexity however, flexible capacity expands the feasible region for production opportunities, allowing firms to hedge better against future stochastic demand variability.

For the two-product-family PFMS problem with quadratic revenue functions, Figure 1 illustrates how the existence of flexible capacity influences the feasible region for the post-demand-realization production problem. This is explained as follows. For $i = 1, \ldots, k$, let the expected revenue functions satisfy

$$p_i R_{iA}(X_{iA}) = -a_i X_{iA}^2 + \bar{b}_i X_{iA} \quad \text{and} \quad p_i R_{iB}(X_{iB}) = -c_i X_{iB}^2 + \bar{d}_i X_{iB}.$$

Recalling that the variable production costs are C_A and C_B, we let $b_i = \bar{b}_i - C_A$ and $d_i = \bar{d}_i - C_B$ so that the objective function for the two-product-family PFMS problem with quadratic revenue functions is

$$-r_A K_A - r_B K_B - r_{AB} K_{AB} + \sum_{i=1}^{k} [-a_i X_{iA}^2 + b_i X_{iA} - c_i X_{iB}^2 + d_i X_{iB}].$$

Conceptualizing the PFMS problem as a two-stage mathematical program, consider the second stage, the selection of production levels $X_{ij} = Y_{ij} + Z_{ij}$, for $i = 1, \ldots, k$ and $j = A, B$. Let $\hat{X}_{ij}$ be the production level for product j in state i that maximizes profits, assuming no capacity constraints. For the quadratic problem, these production levels are easily derived as $\hat{X}_{iA} = b_i/2a_i$ and $\hat{X}_{iB} = d_i/2c_i$. For each state i, we can plot the point $(b_i/2a_i, d_i/2c_i)$ on a two-dimensional product-space graph like Figure 1, which is divided into six regions that depend upon the values of $\bar{K}_A$, $\bar{K}_B$, and $\bar{K}_{AB}$. This figure is useful for describing how to optimally allocate scarce capacity in high-demand states. Region 1 corresponds to the feasible production region, given the capacity constraints. In the

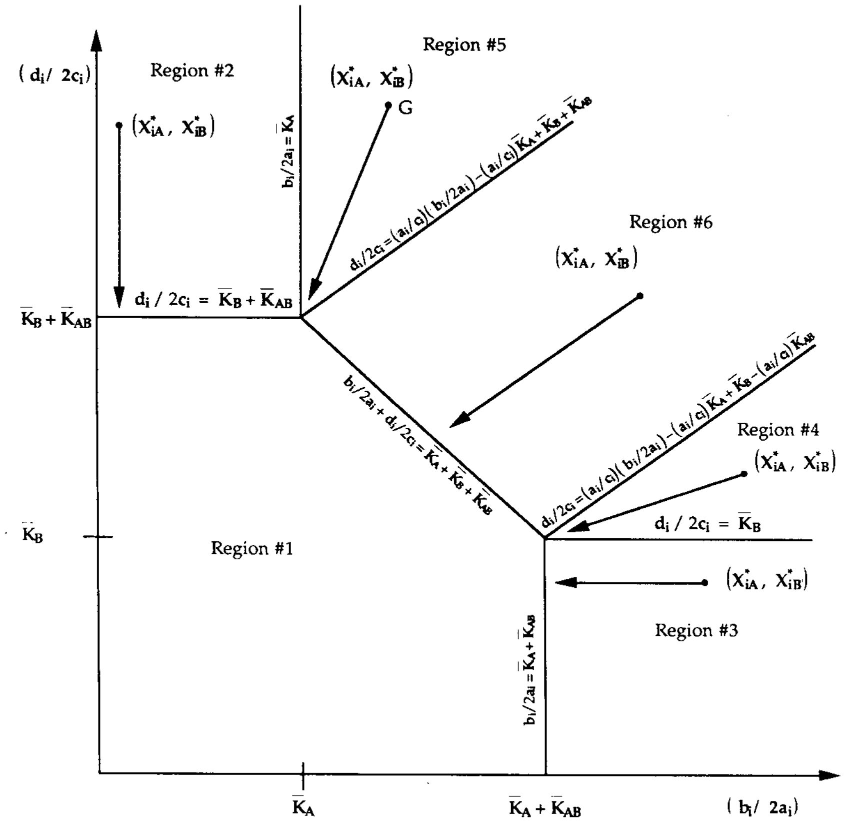

FIGURE 1. The Six Regions into Which the Unconstrained Optimal Production Quantities May Fall.

demand states for which $(\hat{X}_{iA}, \hat{X}_{iB})$ falls in this region, there is no capacity shortfall. If $(\hat{X}_{iA}, \hat{X}_{iB})$ falls in Regions 2, 3, 4, 5 or 6, then capacity is allocated to the products as suggested by the arrows in the figure. For example, suppose that $(\hat{X}_{iA}, \hat{X}_{iB})$ is at the point labeled "G" in the figure. Then following the arrows, we see that the optimal capacity allocation is to allocate all feasible capacity to product B. Production of B is $(\bar{K}_B + \bar{K}_{AB})$ and production of A is $\bar{K}_A$. (See Chakravarty 1987 for a related perspective. See Freund and Find 1986 for further mathematical details.)

One consequence of our model formulation is that the optimal capacity investment and production policy often dictates that one product be manufactured on two different technologies: the dedicated technology and the flexible technology. Mathematically, this result obtains, in part, because for each technology, we have assumed capacity acquisition costs are proportional to the amount of capacity purchased; there is no lump-sum fixed cost to procure any technology. This assumption is clearly unrealistic in some cases. If capacity acquisition costs were concave in the amount of capacity purchased, then, for sufficiently large economies of scale, it would be optimal to acquire either flexible capacity or dedicated capacity, but not both. However, in many cases, even the presence of such economies of scale in capacity procurement would not prevent the optimal policy from dictating that a single product be produced with two different technologies. (This outcome is observed in the electric power generating industry, where firms produce one homogeneous good: electricity; with several different technologies: nuclear, gas, coal, oil, hydroelectric, and wind power.)

For our model, the policy of using two different technologies for producing one type of product has a natural interpretation. For each product, think of demand as broken up into a "base" or certain amount and a "swing" or uncertain amount. Clearly the firm will choose to cover its assured, base demand for each product with the less-costly, dedicated capacity. Some amount of the swing demand for each product, because it is less certain, can be secured by the flexible capacity, which, provided that there is some negative correlation among the products' swing demands, can protect efficiently against the uncertain demand for all of the products.

5. Sensitivity of Solutions to Correlation and Variability in Demand

In this section we explore, with a two-product numerical example, how changes in the probability distributions of demand affect the optimal capacity decisions and the optimal value function. We also explore how the changes in the distribution of demand affects the *threshold cost of flexible capacity*, defined as the largest possible value of the unit cost r_{AB} such that buying a positive amount of flexible capacity is optimal. We focus our inquiry on two issues: (1) How does the correlation between the demand distributions for the two products affect the firm's optimal policies and profits, and (2) how does the riskiness of the probability distributions affect policies and profits?

For our two-product symmetric example, we assume $r_A = r_B = 10$, $r_{AB} = 15$, and that the firm faces quadratic profit functions,

$$b_{iA}X_{iA} - X_{iA}^2 \quad \text{and} \quad b_{iB}X_{iB} - X_{iB}^2,$$

respectively for products A and B. We assume that b_{iA} and b_{iB} can each take on three values: 50, 100, and 150; corresponding to an outcome of low, medium, and high product demand and profitability. Thus we have nine possible states of the world: all of the possible pairs of (b_{iA}, b_{iB}). We represent the probability distributions by the nine-vector $\{p_i; i = 1, \ldots, 9\}$ and with the three-by-three matrix:

	$b_{1B} = 50$	$b_{2B} = 100$	$b_{3B} = 150$
$b_{1A} = 50$	p_1	p_2	p_3
$b_{2A} = 100$	p_4	p_5	p_6
$b_{3A} = 150$	p_7	p_8	p_9 .

We first consider the case where the demands for the two products are perfectly negatively correlated. That is, let $p_1 = p_2 = p_4 = p_6 = p_8 = p_9 = 0$, and parameterize the distribution with the variable p, so that $p_3 = p_7 = p$ and $p_5 = 1 - 2p$. Thus varying p over the range [0, 0.5], traces all of the symmetric, three point probability distributions with mean 100 and correlation coefficient -1. The variable p determines the riskiness of the probability distribution: $p = 0$ represents zero riskiness; $p = 0.5$ represents maximal riskiness. (See Rothschild and Stiglitz 1970, 1971 for motivation and analysis of this definition of riskiness.)

Because of the symmetry between the two products, the optimal solution always has the property that $\bar{K}_A = \bar{K}_B$. For the negative correlation case with $r_{AB} = 15$, Table 1 shows the optimal $\bar{K}_A$, $\bar{K}_{AB}$, and objective function value as a function of p. Figure 2 shows, as a function of p, the threshold value of r_{AB}, labeled $\tilde{r}_{AB}$, at which the optimal quantity of flexible capacity purchased first exceeds zero. For example, when $p = 0.1$, $\bar{K}_{AB} = 0$ if $r_{AB} > 15.555$ and $\bar{K}_{AB} > 0$ if $r_{AB} < 15.555$. From Table 1 we see that over the range of p from 0.1 to 0.5, the optimal quantity of flexible capacity increases in the riskiness of the distribution. It also shows that the flexible capacity is replacing nonflexible capacity over this range (though not one-for-one), and that increased riskiness yields higher expected profits. The benefit from the increased likelihood of the high demand state outweighs the detriment of the companion increase in the likelihood of the low demand state. Figure 2 reinforces the message of Table 1: the threshold value of r_{AB} increases in p. As the level of risk increases, the threshold cost of flexible capacity increases, though not above $r_{AB} = 20$. This is because when $r_{AB} \geq 20 = r_A + r_B$, it is never optimal to invest in flexible capacity, regardless of the level of risk.

TABLE 1
Results for Negative Correlation Case When $r_{AB} = 15$

P	$\bar{K}_A, \bar{K}_B$	$\bar{K}_{AB}$	Optimal Objective Value	$\tilde{r}_{AB}$
0.00	45.00	0.00	4,050	10.00
0.05	46.05	0.00	4,092	12.89
0.10	43.75	6.25	4,141	15.56
0.15	34.52	23.81	4,226	17.94
0.20	29.17	33.33	4,333	20.00
0.25	25.00	40.00	4,450	20.00
0.30	24.17	41.67	4,571	20.00
0.35	23.57	42.86	4,693	20.00
0.40	23.13	43.75	4,816	20.00
0.45	22.78	44.44	4,939	20.00
0.50	22.50	45.00	5,062	20.00

From this example, we see flexible capacity playing the role that we would intuitively expect: Increased riskiness stimulates the need for flexible capacity when demand is negatively correlated between two products.

For the positive correlation example, we assume $p_2 = p_3 = p_6 = p_4 = p_7 = p_8 = 0$, $p_1 = p_9 = p$, and $p_5 = 1 - 2p$. Again, p parameterizes the riskiness of the distribution, but now the two products' demands are perfectly positively correlated. Table 2 summarizes our results, as p ranges from 0 (no risk) to 0.5 (maximum risk). For this example, $\tilde{r}_{AB} = 10$ for all values of $p \in [0, 0.5]$. As long as flexible capacity costs more per unit than the nonflexible capacity, no flexible capacity will be purchased. Increased risk leads to larger purchases of dedicated capacity, but no purchase of flexible capacity. Although perhaps surprising at first, the intuition behind this outcome is quite straightforward: Because the demands for the two products move in lockstep, flexible capacity can only be useful if it can produce one product more cheaply than the dedicated capacity can. There will never be an opportunity to take advantage of the flexibility characteristic of the flexible capacity. Since we assume throughout the paper that $r_{AB} > r_A$ and $r_{AB} > r_B$, flexible capacity will never be acquired when the two products' demands are perfectly positively correlated.

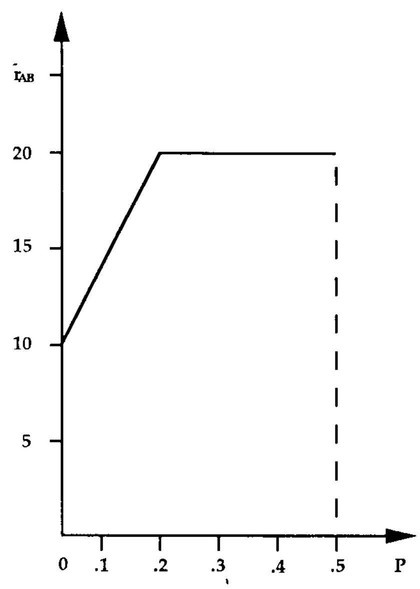

FIGURE 2. Threshold Value of $\tilde{r}_{AB}$, Negative Correlation Case.

TABLE 2

Results for Positive Correlation Case When $r_{AB} = 15$

P	$\bar{K}_A, \bar{K}_B$	$\bar{K}_{AB}$	Optimal Objective Value	$\bar{r}_{AB}$
0.0	45.06	0	4,050	10
0.1	47.22	0	4,139	10
0.2	50.00	0	4,250	10
0.3	58.33	0	4,417	10
0.4	62.50	0	4,625	10
0.5	65.00	0	4,850	10

These two cases illustrate the intuition that the need for flexible capacity increases relative to the level of risk in the presence of negatively correlated demand, and is constant (at level zero) regardless of the level of risk, in the presence of positively correlated demand. Based on these two cases, one might expect that, when the two products' demands are uncorrelated, the optimal quantity of flexible capacity would be nondecreasing in the level of risk. However, this is not always the case, as our third example illustrates.

This final example examines the case where the two products' demand distributions are independent (i.e., zero correlation). We again parameterize the riskiness of the distribution by $p \in [0, 0.5]$, and represent the three-by-three probability matrix by

	$b_{1B} = 50$	$b_{2B} = 100$	$b_{3B} = 150$
$b_{1A} = 50$	p^2	$p(1-2p)$	p^2
$b_{2A} = 100$	$p(1-2p)$	$(1-2p)^2$	$p(1-2p)$
$b_{3A} = 150$	p^2	$p(1-2p)$	p^2 .

Note that for the valid values of p, $0 \le p \le 0.5$, the above distribution has b_A and b_B uncorrelated. Also, $p = 0$ corresponds to zero risk, $p = 0.5$ corresponds to maximum risk, and the level of risk is increasing in p. Table 3 and Figures 3 and 4 summarize the results for this example. From Figure 3 we observe that as a function of p, $\bar{K}_{AB}$ increases from zero at $p = 0$, hits its maximum at $p = 0.33$, and then decreases to zero at $p = 0.5$. From Figure 4, we see that $\bar{r}_{AB} = 10$ when $p = 0$, increases to 18 at $p = 0.2$, and then decreases to 15 at $p = 0.5$.

Although casual intuition suggests that the need for flexible capacity should increase relative to the level of risk (and hence the value of p), this last case shows that this line of reasoning is not always correct. This case shows that the need for flexible capacity is a complex function of the level of demand in each of the future states, and of the probability distribution governing these future states. Indeed, the example points to the fact that the

TABLE 3

Results for Independent Distributions Case When $r_{AB} = 15$

P	$\bar{K}_A, \bar{K}_B$	$\bar{K}_{AB}$	Optimal Objective Value	$\bar{r}_{AB}$
0.00	45.00	0.00	4,050	10.00
0.05	46.05	0.00	4,092	21.75
0.10	47.22	0.00	4,139	15.00
0.15	45.10	6.26	4,196	16.75
0.20	44.28	10.81	4,266	18.00
0.25	45.69	13.10	4,347	17.50
0.30	47.30	14.26	4,436	17.00
0.35	49.01	14.54	4,531	16.50
0.40	54.17	10.42	4,630	16.00
0.45	60.41	4.49	4,737	15.50
0.50	65.00	0.00	4,850	15.00

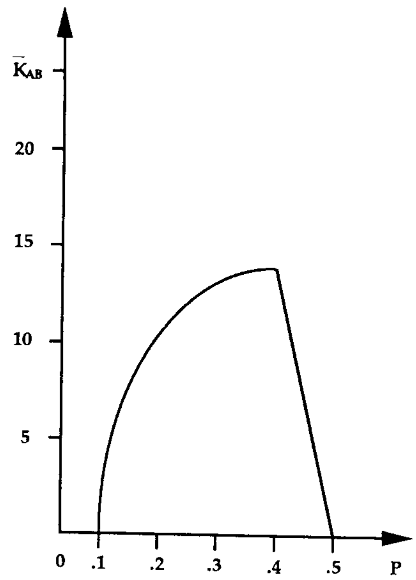

FIGURE 3. Optimal Value of $\bar{K}_{AB}$, Uncorrelated Case.

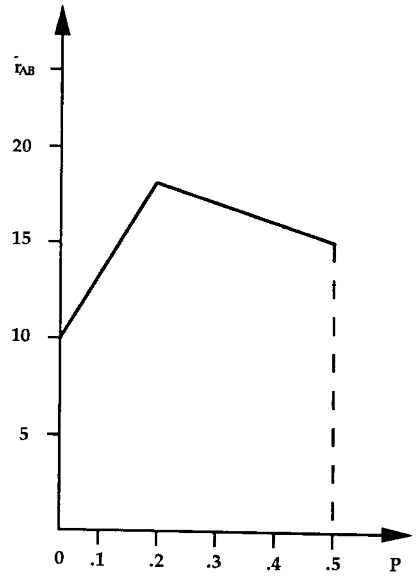

FIGURE 4. Threshold Value of $\bar{r}_{AB}$, Uncorrelated Case.

direction of correlation in demand and the level of risk do not by themselves constitute enough information to predict the sensitivity of the need for flexible capacity to either of these parameters.

6. Concluding Remarks

Our model can be used for several purposes. The formulation and theoretical results are useful for improving one's intuition about the economics of cost/flexibility tradeoffs for the product-flexible manufacturing system investment decision problem. Our results help to explain when the expected return from flexible capacity is likely to exceed its cost (Theorem 1), how optimal profits change when acquisition costs change (Lemma 2), and how optimal capacity investment levels respond to changes in the technology acquisition costs (Theorem 2). In addition, our ripple effect in Theorem 2 illuminates how the existence of flexible capacity links together otherwise independent investment problems.

A second use to which our model can be put is to help shrink the "justification gap" mentioned in our opening paragraphs, by generating estimates of the value of flexibility provided by the flexible technology. A final use for the model is to increase our understanding of the advantages of investing in flexible resources in domains other than the FMS acquisition decision problem. Our model captures cost-flexibility tradeoffs for investment in any type of flexible resource, not only flexible manufacturing systems. In particular, our model can be recast to capture the economics of the common components stocking problem studied by Baker (1985), Baker et al. (1986), Gerchak and Henig (1986) and Gercak et al. (1986). Investing in common components before final demand is known is similar to investing in flexible capacity before demand is known. (Note the likeness between Figure 1 in this paper and Figure 3 in Baker et al. or Figure 6 in Baker.) In ongoing research we are exploring an equivalence between the two problems and the results gained from this insight.

The perpetual adoption of new technology by industry plays a critical role in the economic growth of firms and nations. (See, e.g., Rosenberg 1982 and Schumpeter 1975.) At any point in time, firms face a wide range of options about their investments in technologies and innovations. In the 1980's, computer-controlled, flexible manufacturing systems have emerged as one potentially viable technology for competing in industries that were traditionally characterized by high volume repetition manufacturing, but have more recently been subjected to greater competition and environmental volatility. The aim of this research has been to present a model to help support work in the development of tools to assess the cost/benefit tradeoffs for investment in these product-flexible manufacturing systems.[1]

[1] The authors gratefully acknowledge the helpful comments of Gabriel Bitran, Steve Graves, Tom Magnanti, seminar participants at Stanford, Yale, Columbia, M.I.T., and ORSA/TIMS, two anonymous referees, and an associate editor. The first author received financial support for this work from Cullinet Software, Incorporated and Coopers and Lybrand, Incorporated.

Appendix. Mathematical Properties of the PFMS Model

PROOF OF PROPOSITION 1. 1. The problem is always feasible since $K = 0$, $Y = 0$, $Z = 0$ is feasible. The quantities Y_{ij} and Z_{ij} can be bounded by Q^*_{ij}. Therefore, K_j can be bounded by Q^*_{ij}, and K_F can be bounded nQ^*_{ij}. Because the objective function is continuous, it will attain its optimum in this bounded region.

2. Uniqueness of the production quantities follows from the strict concavity of $R_{ij}(\cdot)$.
3. This result can be proved by contradiction, using the K-K-T conditions.
4. The following counterexample shows the nonuniqueness of $\bar{K}$ when $n = 3$. Let k, the number of possible future states, be three. Let $C_j = 0$ for $j = 1, 2, 3$; let $p_1 = p_2 = p_2 = \frac{1}{3}$, and let the revenue function for each state and product be given by $R_{ij}(X_{ij}) = 3b_{ij} - (3/2)X^2_{ij}$ (a strictly concave quadratic function), where the values of b_{ij} are given below.

Value of b_{ij}

	i		
j	1	2	3
1	17	1	22
2	19	21	1
3	1	19	19

Furthermore, let $r_1 = 10$, $r_2 = 9$, $r_3 = 11$, and $r_F = 15$. Note that $r_F > r_j$ for $j = 1, 2, 3$, and $r_F < r_1 + r_2 + r_3$, as assumed in the model.

The following two solutions are both optimal for this instance of the model. (We leave verification of this claim to the reader.)

Solution #1
$(K_1, K_2, K_3) = (5, 6, 7)$, $K_F = 17$,

Value of Y_{ij}

	i		
j	1	2	3
1	5	1	5
2	6	6	1
3	1	7	7

Value of Z_{ij}

	i		
j	1	2	3
1	8	0	11
2	9	10	0
3	0	7	6

Solution #2
$(K_1, K_2, K_3) = (6, 7, 8)$, $K_F = 15$,

Value of Y_{ij}

	i		
j	1	2	3
1	6	1	6
2	7	7	1
3	1	8	8

Value of Z_{ij}

	i		
j	1	2	3
1	7	0	10
2	8	9	0
3	0	6	5

PROOF OF LEMMA 1. The K-K-T conditions for the PFMS model state that if $\bar{K}_F$, $\bar{K}_j$, $j = 1, \ldots, n$, and $\bar{Y}_{ij}$, $\bar{Z}_{ij}$, $i = 1, \ldots, k$, $j = 1, \ldots, n$ are optimal values for the problem, then there exist nonnegative K-K-T multipliers $\bar{\lambda}_{ij}$, $\bar{s}_{ij}$, $\bar{t}_{ij}$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, $\bar{\gamma}_i$, $i = 1, \ldots, k$, $\bar{u}_j$, $j = 1, \ldots, n$ and $\bar{v}$ such that

(i) $p_i[R'_{ij}(\bar{Y}_{ij} + \bar{Z}_{ij}) - C_j] = \bar{\lambda}_{ij} - \bar{s}_{ij}$, $i = 1, \ldots, k$, $j = 1, \ldots, n$,
(ii) $p_i[R'_{ij}(\bar{Y}_{ij} + \bar{Z}_{ij}) - C_j] = \bar{\gamma}_i - \bar{t}_{ij}$, $i = 1, \ldots, k$, $j = 1, \ldots, n$,
(iii) $r_F = \sum_{i=1}^k \bar{\gamma}_i + \bar{v}$,
(iv) $r_j = \sum_{i=1}^k \bar{\lambda}_{ij} + \bar{u}_j$, $j = 1, \ldots, n$,
(v) $\bar{Y}_{ij}\bar{s}_{ij} = 0$, $\bar{Z}_{ij}\bar{t}_{ij} = 0$, $i = 1, \ldots, k$, $j = 1, \ldots, n$,
(vi) $\bar{v}\bar{K}_F = 0$, $\bar{u}_j\bar{K}_j = 0$, $j = 1, \ldots, n$,
(vii) $\bar{\lambda}_{ij}(\bar{Y}_{ij} - \bar{K}_j) = 0$, $i = 1, \ldots, k$, $j = 1, \ldots, n$,
(viii) $\bar{\gamma}_i(\sum_{j=1}^n \bar{Z}_{ij} - \bar{K}_F) = 0$, $i = 1, \ldots, k$.

Note that because the model is a convex program, and it is possible to find a feasible solution with all constraints satisfied at strict inequalities, then the Slater condition holds, and so conditions (i)–(viii) are both necessary and sufficient for optimality in the model (see Avriel 1976).

Suppose $\bar{K}$, $\bar{Y}$, $\bar{Z}$ are an optimal solution to the model. Let $\bar{\lambda}_{ij}$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, $\bar{\gamma}_i$, $i = 1, \ldots, k$, be defined as in (a) and (b) of Lemma 1.

Define the following multipliers as well:

$$\bar{s}_{ij} = p_i[R'_{ij}(\bar{Y}_{ij} + \bar{Z}_{ij}) - C_j]^-, \qquad i = 1, \ldots, k, \qquad j = 1, \ldots, n,$$

$$\bar{t}_{ij} = \bar{\gamma}_i - p_i[R'_{ij}(\bar{Y}_{ij} + \bar{Z}_{ij}) - C_j], \qquad i = 1, \ldots, k, \qquad j = 1, \ldots, n,$$

$$\bar{u}_j = r_j - \sum_{i=1}^{k} \bar{\lambda}_{ij}, \ j = 1, \ldots, n,$$

$$\bar{v} = r_F - \sum_{i=1}^{k} \bar{\gamma}_i.$$

We must show that all of these multipliers, namely the $\bar{\lambda}_{ij}$, $\bar{\gamma}_i$, $\bar{s}_{ij}$, $\bar{t}_{ij}$, $\bar{u}_j$, $\bar{v}$, satisfy conditions (i)–(viii) above. All multipliers are nonnegative, with the possible exception of $\bar{u}_j$ and $\bar{v}$. Also, K-K-T conditions (i)–(iv) are clearly satisfied. It remains to show that $\bar{u}_j \geq 0$, $\bar{v} \geq 0$, and the complementary slackness conditions (v)–(viii) are satisfied.

Let $\tilde{\lambda}_{ij}$, $\tilde{s}_{ij}$, $\tilde{t}_{ij}$, $\tilde{\gamma}_i$, $\tilde{u}_j$, and $\tilde{v}$ be any optimal K-K-T multipliers. Then, because $(\bar{Y}_{ij} + \bar{Z}_{ij})$ is unique, we must have $\tilde{\lambda}_{ij} - \tilde{s}_{ij} = \bar{\lambda}_{ij} - \bar{s}_{ij}$; and by definition of $\bar{\lambda}_{ij}$ and $\bar{s}_{ij}$, then $\tilde{\lambda}_{ij} \geq \bar{\lambda}_{ij}$, $\tilde{s}_{ij} \geq \bar{s}_{ij}$. Similarly, we can show that $\tilde{\gamma}_i \geq \bar{\gamma}_i$, $\tilde{t}_{ij} \geq \bar{t}_{ij}$. Thus since $\bar{Y}_{ij}\tilde{s}_{ij} = 0$, so must $\bar{Y}_{ij}\bar{s}_{ij} = 0$, since $\bar{s}_{ij} \leq \tilde{s}_{ij}$. Similarly, complementary slackness conditions (v), (vii), and (viii) hold. Also $\tilde{u}_j = r_j - \sum_{i=1}^k \tilde{\lambda}_{ij} \leq r_j - \sum_{i=1}^k \bar{\lambda}_{ij} = \bar{u}_j$, so $\bar{u}_j \geq 0$, and by parallel logic, $\bar{v} \geq 0$ as well. It only remains to show that condition (vi) holds.

If $\bar{K}_j = 0$, then $\bar{u}_j\bar{K}_j = 0$. So suppose $\bar{K}_j > 0$. Then $\tilde{u}_j = 0$. If $\bar{u}_j > 0$, then $\tilde{\lambda}_{ij} > \bar{\lambda}_{ij}$ for some i. For that i, $\tilde{s}_{ij}$

> 0, since $\bar{\lambda}_{ij} - \bar{s}_{ij} = \bar{\lambda}_{ij} - \bar{s}_{ij}$. Hence $\bar{Y}_{ij} = 0$. But $\bar{\lambda}_{ij}(\bar{Y}_{ij} - \bar{K}_j) = 0$, whereby $\bar{K}_j = 0$, a contradiction. Thus $\bar{u}_j = 0$. By similar logic, we derive that if $\bar{K}_F > 0$, then $\bar{v} = 0$.

PROOF OF THEOREM 1. The optimality conditions for the subproblem $P(j)$ state that there exist nonnegative multipliers $\bar{\lambda}_{ij}$, $\bar{s}_{ij}$, $\bar{u}_j$ satisfying

$$r_j = \sum_{i=1}^{k} \bar{\lambda}_{ij} + \bar{u}_j,$$

$$p_i(R'_{ij}(\bar{Y}_{ij}) - C_j) = \bar{\lambda}_{ij} - \bar{s}_{ij},$$

$$\bar{s}_{ij}\bar{Y}_{ij} = 0, \qquad \bar{\lambda}_{ij}(\bar{Y}_{ij} - \bar{K}_j) = 0, \qquad \text{and} \qquad \bar{u}_j\bar{K}_j = 0,$$

whenever $\bar{K}_j$, $\bar{Y}_{ij}$, $i = 1, \ldots, k$, solve $P(j)$, $j = 1, \ldots, n$. Furthermore, it is easy to show that the optimal solution $\bar{K}_j$, $\bar{Y}_{ij}$, $i = 1, \ldots, k$ is unique for each subproblem, $j = 1, \ldots, n$. Let us define $\bar{\gamma}_i = \max_j(\bar{\lambda}_{ij})$ and define $\bar{t}_{ij} = \bar{\gamma}_i - \bar{\lambda}_{ij}$. Then these multipliers, together with $\bar{Z}_{ij} = 0$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, $\bar{K}_F = 0$, satisfy the optimality conditions (i)–(viii) of the PFMS problem, so long as $\sum_{i=1}^{k} \bar{\gamma}_i \le r_F$, i.e., so long as $\sum_i p_i \max_j(\bar{\lambda}_{ij}) \le r_F$. This shows part (b).

The proof of part (a) follows from the above remarks and the uniqueness of the $\bar{\gamma}_i$, $i = 1, \ldots, k$, given in the proof of Lemma 1.

PROOF OF LEMMA 2. We will actually demonstrate a stronger result that extends the well known envelope theorem (see Varian 1978) to the nondifferentiable case, and which will imply the conclusions of Lemma 2 as a special case. Consider the program:

$$P(r): \quad \text{maximize} \quad -rx + f(x)$$
$$\text{subject to} \quad x \in C,$$

where $f(\cdot)$ is a concave function of x, and x is an n-vector, and $C \subset R^n$ is a closed nonempty convex set. Let $d(r)$ be the optimal value of this program parameterized over the n-vector r. The PFMS problem is a special case of this problem, with $x = (K_F, K_1, \ldots, K_n, Y_{11}, \ldots, Y_{kn}, Z_{11}, \ldots, Z_{kn})$, etc. We assume that $P(r)$ attains its optimal value, for any value of r. This is clearly the case, by Proposition 1, for the PFMS problem.

PROOF OF LEMMA 2, PART 1. It suffices to show that $d(r)$ is a convex function. Let r_1 and r_2 be distinct values of r, and let $\bar{r} = pr_1 + qr_2$, where $p \ge 0$, $q \ge 0$, and $p + q = 1$. We must show that $d(\bar{r}) \le pd(r_1) + qd(r_2)$. Let $\bar{x}$ be an optimal solution to $P(\bar{r})$. Thus $d(\bar{r}) = -\bar{r}\bar{x} + f(\bar{x})$. Because $\bar{x}$ is feasible, i.e. $\bar{x} \in C$, we have

$$d(r_1) \ge -r_1\bar{x} + f(\bar{x}), \qquad d(r_2) \ge -r_2\bar{x} + f(\bar{x}).$$

Combining these two inequalities with weights p and q, we obtain $pd(r_1) + qd(r_2) \ge -(pr_1 + qr_2)\bar{x} + f(\bar{x}) = d(\bar{r})$.

PROOF OF LEMMA 2, PARTS 3 AND 4. We first show that $-\bar{x}$ is a subgradient of $d(r)$ at $\bar{r}$. Note that, for any r, $d(r) \ge -r\bar{x} + f(\bar{x}) = -r\bar{x} + d(\bar{r}) + \bar{r}\bar{x} = d(\bar{r}) + (-\bar{x})(r - \bar{r})$. Thus $-\bar{x}$ is a subgradient of $d(r)$ at $r = \bar{r}$. The proof will be complete if we can show the converse: if $-\bar{x}$ is a subgradient of $d(r)$ at $\bar{r}$, then $\bar{x}$ is an optimal solution to $P(\bar{r})$. If this converse is true, then the uniqueness of the optimal solution will mean the subgradient $-\bar{x}$ is unique. Hence $-\bar{x}$ is the *gradient* of $d(r)$, and $\partial d(\bar{r})/dr_j = -\bar{x}_j$ for all j.

We now proceed to prove the converse. Let us define

$$g(x) = \begin{cases} -f(x) & \text{if} \quad x \in C, \\ +\infty & \text{if} \quad x \notin C, \end{cases}$$

Then $g(x)$ is a closed convex function. Its convex conjugate (see, e.g. Avriel 1976) is given by $g^*(r) = \max_x \{rx - g(x)\}$, and it is easy to verify that $d(r) = g^*(-r)$. Suppose $-\bar{x}$ is a subgradient of $d(r)$. Then $\bar{x}$ is a subgradient $g^*(-r)$, and by conjugate duality (see Avriel, Theorem 5.4), $-r$ is a subgradient of $g(x)$ at $x = \bar{x}$. Thus $g(\bar{x}) \ge g(\bar{x}) - r(x - \bar{x})$ for all $x \in C$. Thus $\bar{x} \in C$, so this inequality can be stated $-f(x) \ge -f(\bar{x}) - r(x - \bar{x})$ for all $x \in C$. Thus $-r\bar{x} + f(\bar{x}) \ge f(x) - rx$ for all $x \in C$. Thus $\bar{x}$ solves $P(\bar{r})$.

PROOF OF LEMMA 2, PART 2. This follows directly from the fact that $z^*(r)$ is convex, and all of its subgradients at all values of r are nonpositive, by part 4 of the Lemma.

PROOF OF THEOREM 2. For the case when all the revenue functions are quadratic, this theorem is proved in Freund and Fine (1986). Its extension to nonquadratic, but strictly concave, revenue functions makes sense on intuitive grounds and is a straightforward extension of the quadratic case. Herein, we outline how the proof in Freund and Fine (1986) can be modified for general strictly concave revenue functions.

Theorem 2 is concerned with local behavior of the optimal capacity function $\bar{K}(r)$, for $r \ge 0$. The region $\{r \in R^3 \mid r \ge 0\}$ can be partitioned into a finite number of dense regions, such that in each region a certain set of the constraints in the PFMS problem is active, i.e., binding, at the optimal solution. The interiors of these regions will be dense in R^3, and for any point $\bar{r}$ in the interior of one of these regions, we can compute the matrix M of optimal capacity/cost partial deviatives exactly as in the quadratic case in Freund and Fine (1986), replacing the values given in Table 2, Freund and Fine, of $-2a_i$ by $-R'_{i1}(\bar{Y}_{i1} + \bar{Z}_{i1})$, etc. This shows how to prove Theorem 3 for all values of r in the interior of the different optimal active set regions. However, it is straightforward to show that $K(r)$ is a continuous function, and so by continuity the result is true for all values of r.

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