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Projective transformations for interior-point algorithms, and a superlinearly convergent algorithm for the w-center problem

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The purpose of this study is to broaden the scope of projective transformation methods in mathematical programming, both in terms of theory and algorithms. We start by generalizing the concept of the analytic center of a polyhedral system of constraints to the w-center of a polyhedral system, which stands for weighted center, where there is a positive weight on the logarithmic barrier term for each inequality constraint defining the polyhedron X. We prove basic results regarding contained and containing ellipsoids centered at the w-center of the system X. We next shift attention to projective transformations, and we exhibit an elementary projective transformation that transforms the polyhedron X to another polyhedron Z, and that transforms the current interior point to the w-center of the transformed polyhedron Z. We work throughout with a polyhedral system of the most general form, namely both inequality and equality costraints.

This theory is then applied to the problem of finding the w-center of a polyhedral system X. We present a projective transformation algorithm, which is an extension of Karmarkar's algorithm, for finding the w-center of the system X. At each iteration, the algorithm exhibits either a fixed constant objective function improvement, or converges superlinearly to the optimal solution. The algorithm produces upper bounds on the optimal value at each iteration. The direction chosen at each iteration is shown to be a positively scaled Newton direction. This broadens a result of Bayer and Lagarias regarding the connection between projective transformation methods and Newton's method. Furthermore, the algorithm specializes to Vaidya's algorithm when used with a line-search, and so shows that Vaidya's algorithm is superlinearly convergent as well. Finally, we show how the algorithm can be used to construct well-scaled containing and contained ellipsoids at near-optimal solutions to the w-center problem.

Key words: Analytic center, w-center, projective transformation, linear program, ellipsoid, barrier penalty, Newton method.

1. Introduction

The w-center of a polyhedral system

In [16], Karmarkar simultaneously introduced ideas regarding the center of a polyhedral system, a projective transformation that centers a given point, and a

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linear programming algorithm that uses this methodology to decrease a potential function involving an objective function component and a centering component. Karmarkar's ideas have since been generalized along a number of lines, both theoretical and computational. Herein, we expand on Karmarkar's methodology in at least two ways. First we analyze the w-center of a polyhedron system $X = \{x \in \mathbb{R}^n \mid Ax \leq b, Mx = g\}$, defined as the solution $\hat{x}$ to the following optimization problem:

$$\mathrm{P_w}: \quad \text{maximize} \quad \sum_{i=1}^{m} w_i \ln s_i$$

$$\text{subject to} \quad Ax + s = b,$$

$$Mx = g,$$

$$s > 0.$$

Note that $\mathrm{P_w}$ is a generalization of the analytic center problem first analyzed by Sonnevend [24, 25]. This problem has had numerous applications in mathematical programming; see Renegar [20], Gonzaga [14], and Monteiro and Adler [17, 18], among others. Also note that $\mathrm{P_w}$ is defined for the most general polyhedral representation, namely inequality as well as equality constraints of arbitrary form. In $\mathrm{P_w}$, the weights w_i can be arbitrary positive scalars, and for convenience they are normalized so that $\sum_{i=1}^{m} w_i = 1$. Let $\bar{w}$ be the smallest weight, i.e., $\bar{w} = \min_i\{w_i\}$. The main result for the w-center problem is that if $\hat{x}$ is the w-center, then there exist well-scaled contained and containing ellipsoids at $\hat{x}$ as follows. Let $X = \{x \in \mathbb{R}^n \mid Ax \leq b, Mx = g\}$. Then there exist ellipsoids E_{IN} and E_{OUT} centered at $\hat{x}$, for which $E_{\mathrm{IN}} \subset X \subset E_{\mathrm{OUT}}$, and $(E_{\mathrm{OUT}} - \hat{x}) = ((1-\bar{w})/\bar{w})(E_{\mathrm{IN}} - \hat{x})$, i.e., the outer ellipse is a scaled copy of the inner ellipse, with scaling factor $(1-\bar{w})/\bar{w}$. When the weights are identical, $w = (1/m)\mathrm{e}$, and $((1-\bar{w})/\bar{w}) = (m-1)$. Essentially, the scaling factor $(1-\bar{w})/\bar{w}$ is (almost exactly) inversely proportional to the smallest (normalized) weight w_i.

Projective w-centering for polyhedra in arbitrary form

Numerous researchers have extended Karmarkar's projective transformation methodology, and this study broadens this methodology as well. Gay [11] has shown how to apply Karmarkar's algorithm to linear programming problems in standard form (i.e., "$Ax = b, x \geq 0$"), and how to process inequality constraints by implicitly converting them to standard form. Later, Gay [12] shows how to process problems in standard form with upper and lower bounds, as does Rinaldi [22]. Bayer and Lagarias [4] have added to the theoretical foundations for linear programming by showing that for inequality constrained problems, there exists a class of projective transformation for centering a polyhedron about a given point $\bar{x}$. Anstreicher [2] has shown a different methodology for processing linear programming problems in standard form, and in [7] the author gives a simple projective transformation that constructively uses the result of Bayer and Lagarias for linear programming problems with inequality constraints. Even though linear programs in any one form (e.g., standard primal form) can be either linearly or projectively transformed into another form, such transformations can be computationally bothersome and awkward, and lack aesthetic appeal. Herein, we work throughout with the most general polyhedral system, namely $X = \{x \in \mathbb{R}^n \mid Ax \leq b, Mx = g\}$. This system contains all of the above as special cases, without transformations, addition or elimination of variables, etc. In Sections 3 and 4 of this paper, we present an elementary projective transformation that projectively transforms a general polyhedral system

$$X = \{x \in \mathbb{R}^n \mid Ax \leq b, Mx = g)$$

to an equivalent system

$$Z = \{z \in \mathbb{R}^n \mid \tilde{A}z \leq \tilde{b}, Mz = g),$$

and that results in a given point $\bar{x}$ (in the relative interior of X) being the w-center of the polyhedral system Z. The approach taken is based on classical polarity theory for convex sets; see Rockafellar [23] and Grünbaum [15].

A canonical optimization problem

The results on the w-center problem are applied to the following canonical optimization problem:

$$\text{CP:} \quad \underset{x}{\text{minimize}} \quad F(x) = \ln(U - c^{\mathrm{T}}x) - \sum_{i=1}^{m} w_i \ln(b_i - A_i x)$$

$$\text{subject to} \quad Ax + s = b,$$

$$s > 0,$$

$$Mx = g,$$

$$c^{\mathrm{T}}x < U.$$

where $X = \{x \in \mathbb{R}^n \mid Ax \leq b, Mx = g\}$ is given. Note that problem CP has two important special cases: linear programming and the w-center problem itself. If c is the objective function vector of a linear program maximization problem defined on $X = \{x \in \mathbb{R}^n \mid Ax \leq b, Mx = g\}$, and if U is an appropriate upper bound on the optimal objective function value, then CP is just the problem of minimizing Karmarkar's potential function (generalized to nonuniform weights w_i on the constraints). If $c = 0$ and $U = 1$, then CP is just the w-center problem $\mathrm{P_w}$. In Section 5 of this paper, we present a local improvement algorithm for program CP that is analogous to and is a generalization of Karmarkar's algorithm.

An algorithm for the w-center problem

In Sections 5 and 6, the methodology and theory regarding the w-center, projecting to the w-center, and the local improvement algorithm for the canonical optimization problem CP, are applied to an algorithm to solve the w-center problem $\mathrm{P_w}$. Other algorithms for this problem have been developed by Censor and Lent [5] and by Vaidya [28]. We present a projective transformation algorithm for finding the w-center that is an extension of the ideas of Karmarkar's algorithm applied to the program CP.

This algorithm produces upper bounds on the optimal objective value at each iteration, and these bounds are used to prove that the algorithm is superlinearly convergent. We also show that the direction chosen at each iteration is a positively scaled Newton direction. Thus, if the algorithm is augmented with a line-search, it specializes to Vaidya's algorithm. Although Vaidya has shown that his algorithm exhibits linear convergence, our approach and analysis demonstrate that his algorithm is actually superlinearly convergent, verifying a conjecture of Vaidya [29] that his algorithm might exhibit stronger convergence properties. We also show that after a fixed number of iterations of the algorithm, one can construct "well-scaled" containing and contained ellipsoids at the current iterate of the algorithm. If $\boldsymbol{X} = \{x \in \mathbb{R}^n \mid Ax \leq b, Mx = g\}$ is the current iterate, one can easily construct ellipsoids F_{IN} and F_{OUT} centered at $\bar{x}$, with the property that $F_{\mathrm{IN}} \subset \boldsymbol{X} \subset F_{\mathrm{OUT}}$, and $(F_{\mathrm{OUT}} = \bar{x}) = (2.9/\bar{w})(F_{\mathrm{IN}} - \bar{x})$. When all weights are identical, then this scale factor is $(2.9m)$ which is $\mathrm{O}(m)$. In general, the order of this scale factor is $\mathrm{O}(1/\bar{w})$, which is the same as for the ellipses E_{IN} and E_{OUT} centered at the optimal solution to $\mathrm{P_w}$, whose scale factor is $(1-\bar{w})/\bar{w} = 1/\bar{w} - 1$.

The paper is organized as follows. Section 2 presents notation, definitions and a characterization of the properties of the w-center. Section 3 presents general results regarding properties of projective transformations of polyhedra. In Section 4, we exhibit an elementary projective transformation for transforming the current point $\bar{x}$ to the w-center of the transformed polyhedral system. In Section 5, we introduce the canonical optimization program CP, and present a projective transformation algorithm for the w-center program $\mathrm{P_w}$. In Section 6, the performance of this algorithm is analyzed, and we demonstrate superlinear convergence. In Section 7, we show that the direction generated by the algorithm at each iterate is a positively-scaled Newton direction, and we discuss consequences of this result.

2. Notation and characterization at the w-center

Throughout this paper, we will be concerned with a system of constraints of the form

$$\begin{aligned} Ax &\leq b, \\ Mx &= g, \end{aligned} \tag{2.1}$$

where A is $m \times n$, M is $k \times n$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $g \in \mathbb{R}^k$. One can think of the constraint system as given by the data (A, b, M, g), and so we denote

$$H(\boldsymbol{X}) = (A, b, M, g) \tag{2.2}$$

as the symbolic representation of the constraint system of (2.1). (One can think of "$H(\boldsymbol{X})$" as standing for hyperplanes and halfspaces.) In many contexts, however, it will be particularly convenient to represent the polyhedron determined by all solutions x of (2.1) and so we write

$$\boldsymbol{X} = \{x \in \mathbb{R}^n \mid Ax \leqslant b,\ Mx = g\}. \tag{2.3}$$

For convenience we assume that A has rank n and M has rank k, and so $m \geqslant n$ and $k \leqslant n$.

If $\boldsymbol{X}$ is given, we denote the slack space of $\boldsymbol{X}$ by

$$\boldsymbol{S} = \{s \in \mathbb{R}^m \mid s \geqslant 0,\ s = b - Ax \text{ for some } x \text{ satisfying } Mx = g\}, \tag{2.4}$$

i.e., $\boldsymbol{S}$ is the space of all slack vectors $s = b - Ax$ of the constraint system $\boldsymbol{X}$. We say $\boldsymbol{X}$ has an interior if and only if there exists x for which $Ax < b$ and $Mx = g$, and we write $\operatorname{int} \boldsymbol{X} \neq \emptyset$. Likewise, if there is a vector $s \in \boldsymbol{S}$ for which $s > 0$, then $\boldsymbol{S}$ has an interior and we write $\operatorname{int} \boldsymbol{S} \neq \emptyset$. Obviously $\operatorname{int} \boldsymbol{X} \neq \emptyset$ if and only if $\operatorname{int} \boldsymbol{S} \neq \emptyset$.

Also, we use the following standard notation for diagonal matrices: if $w, s, \bar{s}$, are vectors in $\mathbb{R}^m$, then $W, S, \bar{S}$ denote the diagonal matrices whose diagonal entries correspond to the vectors $w, s, \bar{s}$. Let $e = (1, \ldots, 1)^{\mathrm{T}}$ denote the column of ones of appropriate dimension. Let e_i denote the ith unit vector.

Let w be a vector in $\mathbb{R}^m$ for which $w > 0$ and w has been normalized so that

$$e^{\mathrm{T}} w = 1, \quad w > 0. \tag{2.5}$$

Consider the problem

$$\begin{aligned} \mathrm{P_w}: \quad & \text{maximize} \quad F_w(x) = \sum_{i=1}^{m} w_i \ln(b_i - A_i x) \\ & \text{subject to} \quad Ax + s = b, \\ & \qquad\qquad\quad Mx = g, \\ & \qquad\qquad\quad s > 0. \end{aligned} \tag{2.6}$$

This problem is a (weighted) generalization of the analytic center problem, posed by Sonnevend [22, 23], and used extensively in interior point algorithms for solving linear programming problems; see Renegar [20], Gonzaga [14], and Monteiro and Adler [17, 18], among others.

Under the assumption that $\boldsymbol{X}$ is bounded and $\operatorname{int} \boldsymbol{X} \neq \emptyset$, then $\mathrm{P_w}$ will have a unique solution, $\bar{x}$, which we call the *w-center* of the constraint system $H(\boldsymbol{X})$. The Karush–Kuhn–Tucker (K–K–T) conditions are necessary and sufficient for optimality in $\mathrm{P_w}$, and thus $\bar{x}$ is the w-center of $H(\boldsymbol{X})$ if and only if $\bar{x}$ satisfies

$$A\bar{x} + \bar{s} = b, \tag{2.7a}$$

$$M\bar{x} = g, \tag{2.7b}$$

$$\bar{s} > 0, \tag{2.7c}$$

$$w^{\mathrm{T}} \bar{S}^{-1} A = \bar{\pi}^{\mathrm{T}} M \quad \text{for some } \bar{\pi} \in \mathbb{R}^k. \tag{2.7d}$$

Let $\bar{w}$ denote the smallest component of w, i.e.,

$$\bar{w} = \min\{w_1, \ldots, w_m\}, \tag{2.8a}$$

and define

$$r = \sqrt{\frac{\bar{w}}{1-\bar{w}}}, \qquad R = \sqrt{\frac{1-\bar{w}}{\bar{w}}}. \tag{2.8b}$$

Generalizing Sonnevend [22, 23], we have the following properties of the w-center of $H(\boldsymbol{X})$, that characterize inner and outer ellipsoids centered at $\bar{x}$.

Theorem 2.1. *Let $\boldsymbol{X} = \{x \in \mathbb{R}^n \mid Ax \leq b, Mx = g\}$, let $\bar{x}$ be the w-center of $H(\boldsymbol{X})$, and let $\bar{s} = b - A\bar{x}$. Let*

$$E_{\text{IN}} = \{x \in \mathbb{R}^n \mid Mx = g, (x-\bar{x})^{\mathrm{T}} A^{\mathrm{T}} \bar{S}^{-1} W \bar{S}^{-1} A(x-\bar{x}) \leq r^2\}$$

and

$$E_{\text{OUT}} = \{x \in \mathbb{R}^n \mid Mx = g, (x-\bar{x})^{\mathrm{T}} A^{\mathrm{T}} \bar{S}^{-1} W \bar{S}^{-1} A(x-\bar{x}) \leq R^2\},$$

where r and R are defined in (2.8). *Then*

$$E_{\text{IN}} \subset \boldsymbol{X} \subset E_{\text{OUT}}.$$

Before proving this theorem, we make the following remark.

Remark 2.1. $(E_{\text{OUT}} - \bar{x}) = (R/r)(E_{\text{IN}} - \bar{x})$, i.e., the outer ellipse is a scaled copy of the inner ellipse, with scaling factor $R/r = (1-\bar{w})/\bar{w}$. If $w = (1/m)e$, then $\bar{w} = 1/m$, and so the scaling factor is $R/r = (m-1)$.

The proof of Theorem 2.1 is aided by the following three propositions:

Proposition 2.1. *If $\bar{x}$ is the w-center of $H(\boldsymbol{X})$, and $\bar{s} = b - A\bar{x}$, then the set $\boldsymbol{S}$ defined in* (2.4) *is contained in the simplex $\Delta = \{s \in \mathbb{R}^m \mid s \geq 0, w^{\mathrm{T}} \bar{S}^{-1} s = 1\}$.*

Proof. If $s \in \boldsymbol{S}$, then $w^{\mathrm{T}} \bar{S}^{-1} s = w^{\mathrm{T}} \bar{S}^{-1}(b - Ax)$ for some $x \in \boldsymbol{X}$, and so $w^{\mathrm{T}} \bar{S}^{-1} s = w\bar{S}^{-1}(\bar{s} + A\bar{x} - Ax) = w^{\mathrm{T}} \bar{S}^{-1} \bar{s} + w^{\mathrm{T}} \bar{S}^{-1} A(\bar{x} - x)$. From (2.7d), this latter expression equals $w^{\mathrm{T}} \bar{S}^{-1} \bar{s} + \bar{\pi}^{\mathrm{T}} M(\bar{x} - x) = w^{\mathrm{T}} \bar{S}^{-1} \bar{s} = w^{\mathrm{T}} e = 1$, since $M(x - \bar{x}) = g - g = 0$. □

Proposition 2.2. *Suppose $v \in \mathbb{R}^m$, v satisfies $w^{\mathrm{T}} v = 0$, and $v^{\mathrm{T}} W v \leq r^2$, where w and r satisfy* (2.5) *and* (2.8). *Then $|v_i| \leq 1$ for each $i = 1, \ldots, m$.*

Proof. It suffices to show that $v_i \leq 1$, $i = 1, \ldots, m$. For each i, consider the program

$$\begin{aligned} &\text{maximize} && v_i \\ &\text{subject to} && v^{\mathrm{T}} W v \leq w_i/(1-w_i), && (\alpha) \\ & && w^{\mathrm{T}} v = 0. && (\beta) \end{aligned}$$

The optimal solution to this program is: $v^* = (1/(1-w_i))(-w_ie+e_i)$, with K-K-T multipliers, $\alpha=(1-w_i)/(2w_i)$ and $\beta=1$, which satisfy the K-K-T conditions, $e_i=2\alpha Wv+\beta w$. Notice that $v_i^*=1$. Thus if $v^{\mathrm{T}}Wv\leq r^2\leq w_i/(1-w_i)$ and $w^{\mathrm{T}}v=0$ then $v_i\leq 1$. □

Proposition 2.3. *Let $\bar{x}$ be the w-center of $H(\boldsymbol{X})$ and let $\bar{s}=b-A\bar{x}$. If $s\in\mathbb{R}^m$ satisfies $w^{\mathrm{T}}\bar{S}^{-1}s=1$ and $(s-\bar{s})^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}(s-\bar{s})\leq r^2$, then $0\leq s_i\leq 2\bar{s}_i$, $i=1,\dots,m$.*

Proof. Let s be as given in the proposition. Let $v=\bar{S}^{-1}(s-\bar{s})$. Then v satisfies the hypotheses of Proposition 2.2, and hence $|v_i|\leq 1$, $i=1,\dots,m$. Thus $0\leq s_i\leq 2\bar{s}_i$, $i=1,\dots,m$. □

Proof of Theorem 2.1. We first prove that $\boldsymbol{X}\subset E_{\mathrm{OUT}}$. By Proposition 2.1, $\boldsymbol{S}\subset\Delta$. The extreme points of Δ are $(\bar{s}_i/w_i)e_i$, $i=1,\dots,m$. Note that each extreme point satisfies $((\bar{s}_i/w_i)e_i-\bar{s})^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}((\bar{s}_i/w_i)e_i-\bar{s})=(1-w_i)/w_i\leq R^2$. Thus, because Δ is a convex set, every $s\in\boldsymbol{S}$ satisfies $(s-\bar{s})^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}(s-\bar{s})\leq R^2$. But $(s-\bar{s})=-A(x-\bar{x})$, so $(x-\bar{x})A^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}A(x-\bar{x})\leq R^2$. This shows that $\boldsymbol{X}\subset E_{\mathrm{OUT}}$.

We next show that $E_{\mathrm{IN}}\subset\boldsymbol{X}$. Let $x\in E_{\mathrm{IN}}$, and let s be the slack corresponding to x, i.e., $s=b-Ax$. Then $(s-\bar{s})^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}(s-\bar{s})=(x-\bar{x})A^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}A(x-\bar{x})\leq r^2$. Also, similar to Proposition 2.1, it is straightforward to show that $w^{\mathrm{T}}\bar{S}^{-1}s=1$. Thus by Proposition 2.3, $s\geq 0$. Consequently $Ax\leq b$, and since $x\in E_{\mathrm{IN}}$, $Mx=g$. It follows that $x\in\boldsymbol{X}$. □

The next proposition shows how the w-center can be used to construct an upper bound on the slack $s_i=(b-Ax)_i$ of any constraint of $\boldsymbol{X}$, $i=1,\dots,m$.

Proposition 2.4. *Let $\bar{x}$ be the w-center of $H(\boldsymbol{X})$. For each $i=1,\dots,m$, for any $x\in\boldsymbol{X}$, $(b_i-A_ix)\leq\bar{s}_i/w_i$.*

Proof. For any $x\in\boldsymbol{X}$, let $s=b-Ax$. By Proposition 2.1, $w^{\mathrm{T}}\bar{S}^{-1}s=1$, $s\geq 0$, so $s_i\leq\bar{s}_i/w_i$, i.e., $b_i-A_ix\leq\bar{s}_i/w_i$. □

The last result of this section characterizes the behavior of the weighted-logarithmic function $\sum_{i=1}^m w_i\ln(b_i-A_ix)$ near the w-center $\bar{x}$ of $H(\boldsymbol{X})$. This lemma parallels similar results for the uniformly weighted center in Karmarkar [16] and Vaidya [28].

Lemma 2.1. Let *$\bar{x}$ be the w-center of $H(\boldsymbol{X})$, let $\bar{s}=b-A\bar{x}$, and let $d\in\mathbb{R}^n$ be a direction that satisfies $Md=0$, and $d^{\mathrm{T}}A^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}Ad\leq r^2$. Then for all α satisfying $0\leq\alpha<1$,*

$$\sum_{i=1}^m w_i\ln(b_i-A_i(\bar{x}+\alpha d))\geq\sum_{i=1}^m w_i\ln(\bar{s}_i)-\frac{\alpha^2}{2(1-\alpha)}r^2.$$

Proof. Let $v=\bar{S}^{-1}Ad$. Then $w^{\mathrm{T}}v=w^{\mathrm{T}}\bar{S}^{-1}Ad=\bar{\pi}^{\mathrm{T}}Md=0$ for some $\bar{\pi}\in\mathbb{R}^k$ by (2.7d). Furthermore $v^{\mathrm{T}}Wv\leq r^2$. Thus by Proposition 2.2, $|v_i|\leq 1$, $i=1,\ldots,m$. Therefore

$$\begin{aligned}
&\sum_{i=1}^{m} w_i \ln(b_i - A_i(\bar{x}+\alpha d))\\
&\quad=\sum_{i=1}^{m} w_i \ln(\bar{s}_i(1-\alpha v_i))\\
&\quad=\sum_{i=1}^{m} w_i \ln \bar{s}_i+\sum_{i=1}^{m} w_i \ln(1-\alpha v_i)\\
&\quad\geq\sum_{i=1}^{m} w_i \ln \bar{s}_i+\sum_{i=1}^{m} w_i(-\alpha v_i)-\sum_{i=1}^{m} w_i\frac{(\alpha v_i)^2}{2(1-\alpha)} \quad \text{(by Proposition A.2)}\\
&\quad=\sum_{i=1}^{m} w_i \ln \bar{s}_i-\alpha w^{\mathrm{T}}v-\frac{\alpha^2 v^{\mathrm{T}}Wv}{2(1-\alpha)}\\
&\quad\geq\sum_{i=1}^{m} w_i \ln \bar{s}_i-\frac{\alpha^2}{2(1-\alpha)}r^2. \qquad \square
\end{aligned}$$

Theorem 2.1 characterizes the existence of similar outer and inner ellipsoids at the w-center $\bar{x}$ with a scale ratio of $(1-\bar{w})/\bar{w}$. At points near the center, there also exist such inner and outer ellipsoids; see [8].

3. Projective transformations

Let $\boldsymbol{X}$ be the polyhedron defined by (2.2) or (2.3) and let $\boldsymbol{S}$ be the slack space of $\boldsymbol{X}$ defined in (2.4). This section develops a class of projective transformations of $\boldsymbol{X}$ and $\boldsymbol{S}$ into image sets $\boldsymbol{Z}$ and $\boldsymbol{T}$.

Let $\bar{x}$ satisfy $A\bar{x}<b$ and $M\bar{x}=g$, i.e., $\bar{x}\in\operatorname{int}\boldsymbol{X}$, and let $\bar{s}=b-A\bar{x}$ be the slack vector corresponding to $\bar{x}$. Our interest lies in properties of a projective transformation of $\boldsymbol{X}$ of the form

$$z=g(x)=g_{y,\bar{x}}(x)=\bar{x}+\frac{x-\bar{x}}{1-y^{\mathrm{T}}(x-\bar{x})} \tag{3.1}$$

for a suitable choice of the vector parameter $y\in\mathbb{R}^n$ appearing in the denominator of the transformation. The criterion of suitability that we impose is that the denominator $1-y^{\mathrm{T}}(x-\bar{x})$ remains positive for all $x\in\operatorname{int}\boldsymbol{X}$. If y is chosen so that

$$y\in\operatorname{int} Y_{\bar{x}}=\{y\in\mathbb{R}^n\,|\,y=A^{\mathrm{T}}\bar{S}^{-1}\lambda \text{ for some } \lambda>0 \text{ satisfying } \lambda^{\mathrm{T}}e=1\}, \tag{3.2}$$

then it is elementary to verify that $y^{\mathrm{T}}(x-\bar{x})<1$ for all $x\in\operatorname{int}\boldsymbol{X}$, so that the projective transformation $g(x)$ given in (3.1) is well defined for all $x\in\boldsymbol{X}$. Note that $g(x)$ is more formally denoted as $g_{y,\bar{x}}(x)$ because the transformation is parametrized by y

and $\bar{x}$. Also note that $\bar{x}$ is a fixed point of $g(\cdot)$, i.e., $\bar{x} = g(\bar{x})$. If $x \in \operatorname{int} \boldsymbol{X}$ and $z = g(x)$, then it is straightforward to verify that z satisfies the constraint system

$$\begin{aligned} \tilde{A}z &\leqslant \tilde{b}, \\ Mz &= g, \end{aligned} \tag{3.3}$$

where

$$\tilde{A} = A - \bar{s}y^{\mathrm{T}}, \tag{3.4a}$$

$$\tilde{b} = b - \bar{s}y^{\mathrm{T}}\bar{x}. \tag{3.4b}$$

Analogous to (2.2), (2.3), and (2.4), we thus can define the image constraint set of $g(\cdot)$ as

$$H(\boldsymbol{Z}) = H(\boldsymbol{Z}_{y,\bar{x}}) = (\tilde{A}, \tilde{b}, M, g) = (A - \bar{s}y^{\mathrm{T}}, b - \bar{s}y^{\mathrm{T}}\bar{x}, M, g), \tag{3.5}$$

as a constraint system or

$$\begin{aligned} \boldsymbol{Z} = \boldsymbol{Z}_{y,\bar{x}} &= \{z \in \mathbb{R}^n \mid \tilde{A}z \leqslant \tilde{b}, Mz = g\} \\ &= \{z \in \mathbb{R}^n \mid (A - \bar{s}y^{\mathrm{T}})z \leqslant (b - \bar{s}y^{\mathrm{T}}\bar{x}), Mz = g\}, \end{aligned} \tag{3.6}$$

and the slack space of $\boldsymbol{Z}$ as

$$\boldsymbol{T} = \boldsymbol{T}_{y,\bar{x}} = \{t \in \mathbb{R}^m \mid t \geqslant 0, t = \tilde{b} - \tilde{A}z \text{ for some } z \text{ satisfying } Mz = g\}. \tag{3.7}$$

The inverse of $g(\cdot)$ is given by the function

$$x = h(z) = h_{y,\bar{x}}(z) = g_{y,\bar{x}}^{-1}(z) = \bar{x} + \frac{z - \bar{x}}{1 + y^{\mathrm{T}}(z - \bar{x})}. \tag{3.8}$$

The transformations developed in (3.1)-(3.8) are illustrated in Figure 3.1. Finally, we can extend $g(\cdot)$ and $h(\cdot)$ to the slack spaces $\boldsymbol{S}$ and $\boldsymbol{T}$ as follows. Let

$$(\boldsymbol{X}; \boldsymbol{S}) = \{(x, s) \in \mathbb{R}^n \times \mathbb{R}^m \mid Ax + s = b, s \geqslant 0, Mz = g\}$$

and

$$(\boldsymbol{Z}, \boldsymbol{T}) = \{(z, t) \in \mathbb{R}^n \times \mathbb{R}^m \mid \tilde{A}z + t = \tilde{b}, t \geqslant 0, Mz = g\},$$

and define

$$\begin{aligned} (z, t) &= g_{y,\bar{x}}(x, s) \\ &= \left(\bar{x} + \frac{x - \bar{x}}{1 - y^{\mathrm{T}}(x - \bar{x})}, \frac{s}{1 - y^{\mathrm{T}}(x - \bar{x})}\right) \quad \text{for } (x, s) \in (\boldsymbol{X}, \boldsymbol{S}), \end{aligned} \tag{3.9a}$$

$$\begin{aligned} (x, s) &= h_{y,\bar{x}}(z, t) \\ &= \left(\bar{x} + \frac{z - \bar{x}}{1 + y^{\mathrm{T}}(z - \bar{x})}, \frac{t}{1 + y^{\mathrm{T}}(z - \bar{x})}\right) \quad \text{for } (z, t) \in (\boldsymbol{Z}, \boldsymbol{T}). \end{aligned} \tag{3.9b}$$

To formally identify the properties of the transformation $g(\cdot) = g_{y,\bar{x}}(\cdot)$, we consider separately the cases when $\boldsymbol{X}$ is bounded and unbounded.

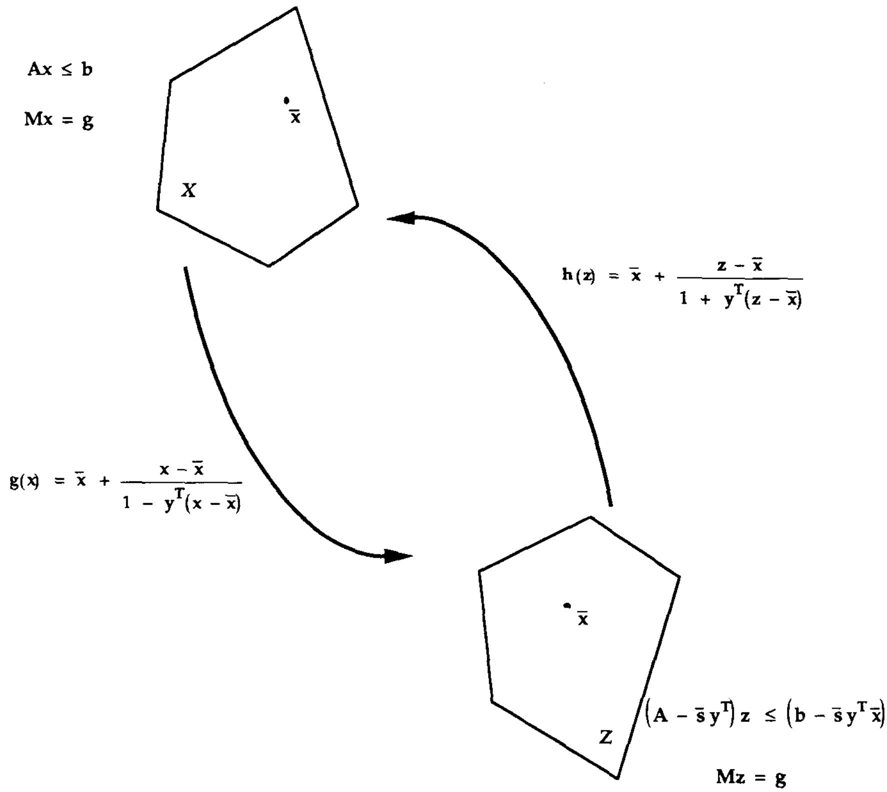

Fig. 3.1. Projective transformation.

Lemma 3.1. *Let $H(X)$ and X be given by* (2.2) *and* (2.3), *and suppose X is bounded. Let $\bar{x} \in \operatorname{int} X$ be given, let $\bar{s} = b - A\bar{x}$, and let y, $g(\cdot)$, $h(\cdot)$, Z, and T satisfy* (3.2)–(3.9). *Then*:

(i) *$g(\cdot)$ maps X onto Z and S onto T.*

(ii) *$h(\cdot)$ maps Z onto X and T onto S.*

(iii) *X and Z are the same combinatorial type, and $g(\cdot)$ maps faces of X onto corresponding faces of Z.*

Proof. It suffices to show that (i) $y^{\mathrm{T}}(x-\bar{x})<1$ for all $x \in X$ and (ii) $-y^{\mathrm{T}}(z-\bar{x})<1$ for all $z \in Z$.

(i) Suppose $x \in X$ and $y^{\mathrm{T}}(x-\bar{x}) \geq 1$. Then from (3.2),

$$1 \leq y^{\mathrm{T}}(x-\bar{x}) = \lambda^{\mathrm{T}}\bar{S}^{-1}A(x-\bar{x}) = \lambda^{\mathrm{T}}\bar{S}^{-1}(Ax-b+\bar{s}) = \lambda^{\mathrm{T}}\bar{S}^{-1}(Ax-b)+1,$$

and so $\lambda^{\mathrm{T}}\bar{S}^{-1}(Ax-b) \geq 0$. Thus $Ax=b$, since $\lambda>0$ and $Ax \leq b$.
Therefore $v=\bar{x}-x$ satisfies $Av=-\bar{s}<0$ and $Mv=0$, and so X is unbounded, which is a contradiction. Therefore $y^{\mathrm{T}}(x-\bar{x})<1$.

(ii) Now suppose $z \in \boldsymbol{Z}$ and $-y^{\mathrm{T}}(z-\bar{x}) \geqslant 1$. Then define $v = z - \bar{x}$ and note that $v \neq 0$. We have

$$Av = Az - A\bar{x} = Az - b + \bar{s} \leqslant Az - b - \bar{s}(y^{\mathrm{T}}(z-\bar{x})) = \tilde{A}z - \tilde{b} \leqslant 0$$

from (3.4)-(3.6). Also $Mv = 0$, so that v is a ray of $\boldsymbol{X}$, which contradicts the boundedness of $\boldsymbol{X}$. □

In the case when $\boldsymbol{X}$ is unbounded, we no longer can guarantee that the projective transformation $g(\cdot)$ is onto and invertible, unless we assume that the system $Ax \leqslant b$ has been enlarged to include a trivial constraint of the form $0^{\mathrm{T}}x \leqslant 1$. We then have:

Lemma 3.2. *Let $H(\boldsymbol{X})$ and $\boldsymbol{X}$ be given by* (2.2) *and* (2.3), *and suppose that the last row of the inequality constraints $Ax \leqslant b$ is of the form $0^{\mathrm{T}}x \leqslant 1$. Let $\bar{x} \in \operatorname{int} \boldsymbol{X}$ be given, let $\bar{s} = b - A\bar{x}$, and let y, $g(\cdot)$, $h(\cdot)$, $\boldsymbol{Z}$ and $\boldsymbol{T}$ satisfy* (3.2)-(3.9). *Then*

(i) *$g(\cdot)$ is well defined for all $x \in \boldsymbol{X}$.*

$g(\cdot)$ maps $\operatorname{int} \boldsymbol{X}$ onto $\operatorname{int} \boldsymbol{Z}$ and $\operatorname{int} \boldsymbol{S}$ onto $\operatorname{int} \boldsymbol{T}$.

$g(\cdot)$ maps bounded faces F of $\boldsymbol{X}$ onto those faces G of $\boldsymbol{Z}$ that do not meet the hyperplane $H = \{z \in \mathbb{R}^n \mid -y^{\mathrm{T}}(z-\bar{x}) = 1\}$.

(ii) *$h(\cdot)$ is well defined for all $z \in \boldsymbol{Z}$, $z \notin H$.*

$h(\cdot)$ maps $\operatorname{int} \boldsymbol{Z}$ onto $\operatorname{int} \boldsymbol{X}$ and $\operatorname{int} \boldsymbol{T}$ onto $\operatorname{int} \boldsymbol{S}$.

$h(\cdot)$ maps faces G of $\boldsymbol{Z}$ that do not meet H onto bounded faces F of $\boldsymbol{X}$.

(iii) *If $z \in \boldsymbol{Z}$ and $z \in H$, then $r = z - \bar{x}$ forms a nontrivial ray of $\boldsymbol{X}$, i.e.,*

$$\{x \in \mathbb{R}^n \mid x = \bar{x} + \lambda(z - \bar{x}) \textit{ for some } \lambda \geqslant 0\} \subset \boldsymbol{X}.$$

Proof. (i) Let $x \in \boldsymbol{X}$, and let $s = b - Ax$. Then $s \geqslant 0$, and from (3.2), $y^{\mathrm{T}}(x-\bar{x}) = \lambda^{\mathrm{T}}\bar{S}^{-1}(\bar{s}-s) = 1 - \lambda^{\mathrm{T}}\bar{S}^{-1}s < 1$, because the last constraint of $Ax \leqslant b$ is $0^{\mathrm{T}}x \leqslant 1$, $\bar{s}_m = s_m = 1$, and $\lambda > 0$. Thus $g(\cdot)$ is well-defined for all $x \in \boldsymbol{X}$. If $z = g(x)$, then it remains to show that $-y^{\mathrm{T}}(z-\bar{x}) < 1$. We have

$$-y^{\mathrm{T}}(z-\bar{x}) = \frac{-y^{\mathrm{T}}(x-\bar{x})}{1 - y^{\mathrm{T}}(x-\bar{x})} < 1,$$

because $y^{\mathrm{T}}(x-\bar{x}) < 1$.

(ii) If $z \in \boldsymbol{Z}$ then the last constraint of $\tilde{A}z \leqslant \tilde{b}$ is $-y^{\mathrm{T}}z \leqslant 1 - y^{\mathrm{T}}\bar{x}$, from (3.4). If $z \notin H$, then $-y^{\mathrm{T}}(z-\bar{x}) < 1$, and so $h(z) = g^{-1}(z)$ is well-defined.

(iii) Suppose $z \in \boldsymbol{Z} \cap H$. Let $r = z - \bar{x}$. Then $Mr = 0$. Also $(A - \bar{s}y^{\mathrm{T}})z \leqslant b - \bar{s}y^{\mathrm{T}}\bar{x}$ and $y^{\mathrm{T}}(z-\bar{x}) = -1$ implies $Az \leqslant b + \bar{s}y^{\mathrm{T}}(z-\bar{x}) = b - \bar{s} = A\bar{x}$, and so $A(z-\bar{x}) \leqslant 0$, i.e., $Ar \leqslant 0$. Finally, notice that $r \neq 0$ (otherwise $y^{\mathrm{T}}(z-\bar{x}) = 0$). Thus

$$\{x \in \mathbb{R}^n \mid x = \bar{x} + \lambda(z-\bar{x}) \text{ for some } \lambda \geqslant 0\} \subset \boldsymbol{X}. \qquad \square$$

As a corollary to both Lemma 3.1 and 3.2 we have:

Corollary 3.1. *Let X be given by* (2.2) *or* (2.3), *and suppose that X satisfies the following condition*:

(A) *Either X is bounded or the last row of the inequalities $Ax \leqslant b$ is of the form*

$$0^{\mathrm{T}}x \leqslant 1. \tag{3.10}$$

Then the mappings $g(\cdot)$ and $h(\cdot)$ of Lemmas 3.1 *or Lemma* 3.2 *are well-defined for all $x \in \operatorname{int} X$* and *$z \in \operatorname{int} Z$.* □

It can be shown (see [8]) that the projective transformation $g(\cdot)$ is quite general, in that any projective transformation $g(\cdot)$ that leaves $\bar{x}$ fixed and preserves directions from $\bar{x}$ can be written in a form satisfying (3.1) and (3.2). The projective transformation $g(x) = g_{y,\bar{x}}(x)$ can also be developed through convex polarity theory. The set $Y_{\bar{x}}$ of (3.2) is the polar of $(X - \bar{x})$; see Grünbaum [15], and Rockafellar [23]. The set $(Z - \bar{x})$ then is the polar of the translation of $Y_{\bar{x}}$ by y, i.e., $Z = ((X - \bar{x})^{\circ} - y)^{\circ} + \bar{x}$; see [8].

4. Projective transformations to w-center a given interior point

Let X be the constraint system defined by (2.2) or (2.3) and let S be the slack space of X defined in (2.4). Let $\bar{x}$ satisfy $A\bar{x} < b$ and $M\bar{x} = g$, i.e., $\bar{x} \in \operatorname{int} X$, and let $\bar{s} = b - A\bar{x}$ be the slack vector corresponding to $\bar{x}$. Suppose we wish to find a projection parameter $y \in Y_{\bar{x}}$ so that $\bar{x}$ is the w-center of the projectively transformed constraint system $H(Z) = H(Z_{y,\bar{x}})$ under the projective transformation $g(x) = g_{y,\bar{x}}(x)$.

Theorem 4.1. *Let $w > 0$ be an m-vector satisfying $e^{\mathrm{T}}w = 1$. Let $H(X)$ and X be a constraint system of the form* (2.2), (2.3), *let $\bar{x} \in \operatorname{int} X$, $\bar{s} = b - A\bar{x}$, and let*

$$y = A^{\mathrm{T}}\bar{S}^{-1}w. \tag{4.1}$$

Then $y \in \operatorname{int}(Y_{\bar{x}})$ given in (3.2), *and $\bar{x}$ is the w-center of the projectively transformed constraint system $H(Z) = H(Z_{y,\bar{x}})$ given by* (3.3)-(3.6), *under the projective transformation $g(x) = g_{y,\bar{x}}(x)$ of* (3.8)-(3.9).

Proof. By setting $\lambda = w$, we see that $y \in \operatorname{int} Y_{\bar{x}}$. Note that $g(\bar{x}, \bar{s}) = (\bar{x}, \bar{s})$, so that $(\bar{x}, \bar{s}) \in (Z; T)$, i.e., $\tilde{A}\bar{x} + \bar{s} = \tilde{b}$, $M\bar{x} = g$. From (2.7), it remains to show that $w^{\mathrm{T}}\bar{S}^{-1}\tilde{A} = \pi^{\mathrm{T}}M$ for some $\pi \in \mathbb{R}^k$. One has $w^{\mathrm{T}}\bar{S}^{-1}\tilde{A} = w^{\mathrm{T}}\bar{S}^{-1}(A - \bar{s}y^{\mathrm{T}}) = w^{\mathrm{T}}\bar{S}^{-1}(A - \bar{s}w^{\mathrm{T}}\bar{S}^{-1}A) = 0$. Hence, one may take $\pi = 0$, so (2.7) is satisfied, completing the proof. □

Theorem 4.1 is a generalization of a theorem of Lagarias [17] which asserts the existence of a projective transformation that will result in $\bar{x}$ being the w-center of a full-dimensional polytope X in the case of $w = (1/m)e$. Theorem 4.1 covers a general linear system of both inequality and equality constraints, and covers the case of non-uniform weights w. Although the projective transformation $g(x) = g_{y,\bar{x}}(x)$ defined in Theorem 4.1 using (4.1) does not appear to resemble Karmarkar's projective transformation [16] for centering in a simplex, it is shown in [8] that Theorem 4.1 specializes to Karmarkar's projective transformation when viewed in the slack space $\boldsymbol{S}$.

5. A canonical optimization problem, and an algorithm for the w-center problem

In this section, we consider the following canonical optimization problem:

$$\text{CP:} \quad \underset{x}{\text{minimize}} \quad F(x) = \ln(U - c^{\mathrm{T}}x) - \sum_{i=1}^{m} w_i \ln(b_i - A_i x)$$

$$\text{subject to} \quad Ax + s = b, \quad s > 0, \quad Mx = g, \quad c^{\mathrm{T}}x < U. \tag{5.1}$$

The data for the problem is the data for the constraint set $H(\boldsymbol{X}) = (A, b, M, g)$, plus the m-vector of positive weights $w = (w_1, \ldots, w_m)^{\mathrm{T}}$ which satisfy the normalization $e^{\mathrm{T}}w = 1$, plus the data for constraint $c^{\mathrm{T}}x < U$. The general linear programming problem:

$$\text{LP:} \quad \text{maximize} \quad c^{\mathrm{T}}x \quad \text{subject to} \quad Ax \leqslant b, \quad Mx = g, \tag{5.2}$$

can be cast as an instance of CP. By setting c to be the LP objective function vector and U to be an upper bound on the optimal LP objective value, CP becomes the potential function minimization problem for LP, as in Karmarkar [16]. This problem instance has already been treated in [7] and also [8].

The problem of finding the w-center, namely problem $\mathrm{P_w}$ defined in (2.6), is also an instance of CP. By setting

$$c = 0 \quad \text{and} \quad U = 1, \tag{5.3}$$

problem CP specializes to problem $\mathrm{P_w}$. In Sections 6 and 7, we present an analysis of problem $\mathrm{P_w}$ viewed through the canonical optimization problem CP.

Suppose now that we wish to solve CP, and that we have on hand a feasible solution $\bar{x}$ of CP, i.e., $\bar{x} \in \operatorname{int} \boldsymbol{X}$ and $c^{\mathrm{T}}\bar{x} < U$. If $\bar{x}$ happens to be the w-center of $\boldsymbol{X}$, then $\bar{x}$ has optimized the second part of the objective function $F(x)$ of CP. If $\bar{x}$ is not the w-center of $\boldsymbol{X}$, we can perform the projective transformation of Theorem

4.1 in order to ensure that $\bar{x}$ is the w-center of the transformed constraint set $\boldsymbol{Z} = \boldsymbol{Z}_{y,\bar{x}}$ (where $y = A^{\mathrm{T}}\bar{S}^{-1}w$ is given in (4.1)) under the projective transformation $z = g(x) = g_{y,\bar{x}}(x)$ of (3.9). Under this projective transformation, the constraints of $\boldsymbol{X}$ are mapped into the constraints of $\boldsymbol{Z}$, which are given by (3.3) and (3.4). Furthermore, if $\bar{x} \in \boldsymbol{X}$ and x satisfies $c^{\mathrm{T}}x \leq U$, it is then elementary to show $z = g_{y,\bar{x}}(x)$ will satisfy

$$\tilde{c}^{\mathrm{T}}z \leq \tilde{U}, \tag{5.4}$$

where

$$\tilde{c} = c - (U - c^{\mathrm{T}}\bar{x})y, \quad \tilde{U} = U - (U - c^{\mathrm{T}}\bar{x})y^{\mathrm{T}}\bar{x}. \tag{5.5}$$

The next lemma shows that under the projective transformation $g_{y,\bar{x}}(\cdot)$, the program CP is transformed into program

$$\begin{aligned} \widetilde{\mathrm{CP}} = \widetilde{\mathrm{CP}}_{y,\bar{x}}: \quad & \underset{z}{\text{minimize}} \quad \tilde{F}(z) = \ln(\tilde{U} - \tilde{c}^{\mathrm{T}}z) - \sum_{i=1}^{m} w_i \ln(\tilde{b}_i - \tilde{A}_i z) \\ & \text{subject to} \quad \tilde{A}z + t = \tilde{b}, \quad t > 0, \\ & \qquad\qquad\quad Mz = g, \\ & \qquad\qquad\quad \tilde{c}^{\mathrm{T}}z < \tilde{U}, \end{aligned} \tag{5.6}$$

where $\tilde{U}$, $\tilde{c}$ are given by (5.5) and $(\tilde{A}, \tilde{b})$ is given by (3.4).

Lemma 5.1 (equivalence of CP and $\widetilde{\mathrm{CP}}_{y,\bar{x}}$). *Suppose $y \in \operatorname{int} Y_{\bar{x}}$ of* (3.2) *and define the projective transformation $g(\cdot) = g_{y,\bar{x}}(\cdot)$ as in* (3.9a) *and its inverse $h(\cdot) = h_{y,\bar{x}}(\cdot)$ as in* (3.9b). *If $\boldsymbol{X}$ satisfies condition* (A) *of Corollary* 3.1, *then programs* CP *and* $\widetilde{\mathrm{CP}}$ *are equivalent, i.e.:*

(i) *if x is feasible for* CP, *$z = g(x)$ is feasible for* $\widetilde{\mathrm{CP}}$ *and $F(x) = \tilde{F}(z)$.*
(ii) *if z is feasible for* $\widetilde{\mathrm{CP}}$, *$x = h(z)$ is feasible for* CP *and $\tilde{F}(z) = F(x)$.*

Proof. (i) If x is feasible for CP, then $x \in \operatorname{int} \boldsymbol{X}$ and so from Corollary 3.1, $z = g(x)$ is well-defined and $z \in \operatorname{int} \boldsymbol{Z}$. The equality $F(x) = \tilde{F}(z)$ follows by direct substitution. A parallel argument also demonstrates assertion (ii). □

Lemma 5.1 shows that by the projective transformation of Theorem 4.1, we can always reduce CP to the special case in which the current feasible solution $\bar{x}$ is the w-center of the constraint set $\boldsymbol{X}$, and satisfies $c^{\mathrm{T}}x < U$.

We therefore suppose, without loss of generality, that we have on hand a feasible solution $\bar{x}$ of CP, i.e., $\bar{x} \in \boldsymbol{X}$, and $c^{\mathrm{T}}x < U$, and that $\bar{x}$ is the w-center of $\boldsymbol{X}$. Then the inner ellipsoid E_{IN} at the w-center is contained in $\boldsymbol{X}$ (from Theorem (2.1), and $F(x)$ can be improved by optimizing $c^{\mathrm{T}}x$ over the inner ellipsoid E_{IN}. From Theorem 2.1, the problem of finding the direction d that maximizes $c^{\mathrm{T}}(\bar{x} + d)$ over the ellipsoid E_{IN} is

$$\begin{aligned} & \text{maximize} \quad c^{\mathrm{T}}d \\ & \text{subject to} \quad d^{\mathrm{T}}A^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}Ad \leq r^2, \\ & \qquad\qquad\quad Md = 0, \end{aligned} \tag{5.7}$$

where r is defined in (2.8), and $\bar{s} = b - A\bar{x}$. Under the assumption that A and M have full rank, program (5.7) has a unique solution given by

$$\bar{d} = rGc/\sqrt{c^{\mathrm{T}}Gc}, \tag{5.8a}$$

where

$$Q = A^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}A \quad \text{and} \quad G = [Q^{-1} - Q^{-1}M^{\mathrm{T}}(MQ^{-1}M^{\mathrm{T}})^{-1}MQ^{-1}]. \tag{5.8b}$$

It is straightforward to check that G is positive semi-definite, and so $c^{\mathrm{T}}Gc \geq 0$. Furthermore, $c^{\mathrm{T}}Gc = 0$ if and only if c^{T} lies in the row space of M, which implies that $\bar{x}$ solves CP since $\bar{x}$ is the w-center of the system $H(X)$. Therefore, unless $\bar{x}$ solves CP, the denominator of (5.8a) is well-defined and $\bar{d}$ given in (5.8) is the unique solution to program (5.7).

The extent of improvement in optimizing $F(z)$ of (5.1) by moving from $\bar{x}$ in the direction $\bar{d}$ of (5.8) is as follows:

Theorem 5.1 (improvement of CP from the w-center $\bar{x}$). *Suppose $\bar{x}$ is the w-center of X, $\bar{s} = b - A\bar{x}$, and let $\bar{d}$ be the solution to* (5.7) *given in* (5.8). *Define the quantity*

$$\gamma = \frac{c^{\mathrm{T}}\bar{d}}{(U - c^{\mathrm{T}}\bar{x})r^2}. \tag{5.9}$$

Then:

(i) *If $\gamma \geq 1/r^2$, the program* CP *is unbounded from below.*

(ii) *If $\gamma < 1/r^2$, then*

$$F(\bar{x} + \alpha\bar{d}) \leq F(\bar{x}) - r^2\left(\gamma\alpha - \frac{\alpha^2}{2(1-\alpha)}\right)$$

for all $\alpha \in [0, 1)$.

Before proving the theorem, we derive a consequence. The optimal objective value of the inner ellipsoid maximization program (5.7) is $c^{\mathrm{T}}\bar{d}$, and so γ is just a rescaling of this value by the quantity $(U - c^{\mathrm{T}}\bar{x})r^2$. In (ii) of the theorem, the extent of improvement in the objective function CP is proportional to the function

$$f(\alpha) = \gamma\alpha - \frac{\alpha^2}{2(1-\alpha)}. \tag{5.10}$$

The value of α that maximizes $f(\alpha)$ over $\alpha \in [0, 1]$ is

$$\alpha = 1 - \frac{1}{\sqrt{1+2\gamma}}, \tag{5.11}$$

which yields the value of $f(\alpha)$ of

$$k(\gamma) = (1 + \gamma - \sqrt{1+2\gamma}). \tag{5.12}$$

Summarizing, we have:

Corollary 5.1. *If α is given in* (5.11), *then*

$$F(\bar{x} + \alpha\bar{d}) - F(\bar{x}) \leq -r^2(1 + \gamma - \sqrt{1+2\gamma}) = -r^2k(\gamma). \qquad \square \tag{5.13}$$

Proof of Theorem 5.1. (i) Suppose $\gamma \geqslant 1/r^2$. Then define $\hat{\alpha} = (\gamma r^2)^{-1} = (U - c\bar{x})/c^{\mathrm{T}}\bar{d}$, and notice that $\hat{\alpha} \leqslant 1$. Then from (5.9), $c^{\mathrm{T}}\bar{d} \geqslant U - c^{\mathrm{T}}\bar{x}$, i.e., $c^{\mathrm{T}}(\bar{x}+d) \geqslant U$. Thus, as $\alpha \to \hat{\alpha}$, $\ln(U - c^{\mathrm{T}}(\bar{x}+\alpha d)) \to -\infty$. As a consequence of (5.7), (5.8), and Theorem 2.1, $\bar{x}+\alpha\bar{d} \in \boldsymbol{X}$ for all $\alpha \in [0, \hat{\alpha})$. If CP is bounded from below, then $A_i(\bar{x}+\alpha\bar{d}) \to b_i$ for every $i = 1, \ldots, m$, as $\alpha \to \hat{\alpha}$, i.e., $A\bar{d} = \bar{s}$ and $\hat{\alpha} = 1$, which implies that $\boldsymbol{X}$ is unbounded, which in turn implies that the w-center of $H(\boldsymbol{X})$ cannot exist, contradicting the hypothesis of the Theorem. Thus CP is unbounded from below. (ii) Suppose $\gamma < 1/r^2$. Then

$$
\begin{aligned}
&F(\bar{x}+\alpha\bar{d}) - F(\bar{x}) \\
&= \ln\left(\frac{U - c^{\mathrm{T}}(\bar{x}+\alpha\bar{d})}{U - c^{\mathrm{T}}\bar{x}}\right) - \sum_{i=1}^{m} w_i \ln\left(\frac{b_i - A_i(\bar{x}+\alpha\bar{d})}{b_i - A_i\bar{x}}\right) \\
&= \ln(1 - \alpha r^2\gamma) - \sum_{i=1}^{m} w_i \ln(1 - \alpha(\bar{S}^{-1}A\bar{d})_i) \quad \text{(from 5.9)} \\
&\leqslant -r^2\alpha\gamma + \frac{r^2\alpha^2}{2(1-\alpha)} \\
&\qquad \text{(from Proposition A.1 of the Appendix and Lemma 2.1)} \\
&= r^2\left(-\gamma\alpha + \frac{+\alpha^2}{2(1-\alpha)}\right). \qquad \square
\end{aligned}
$$

Lemma 5.1 and Theorem 5.1 suggest the following algorithm for solving CP: At each iteration, CP is projectively transformed to $\widetilde{\mathrm{CP}} = \widetilde{\mathrm{CP}}_{y,\bar{x}}$ of (5.6) where $y = A^{\mathrm{T}}\bar{S}^{-1}w$ (of (4.1)), which transforms the current point $\bar{x}$ to the w-center of the transformed constraint set (Theorem 4.1). Then the algorithm steps a length α in the direction $\bar{d}$ of (5.7)–(5.8) that maximizes the transformed objective function vector $\tilde{c}$ over the inner ellipsoid E_{IN}, where α is given by (5.11). The specialization of this algorithm to solving LP is detailed in [8]. The remainder of this section treats the specialization of this methodology to solve the w-center program $\mathrm{P_w}$.

Recall that program $\mathrm{P_w}$ given by (2.6) is the special case of CP where $c = 0$ and $U = 1$ (5.3). The algorithm for solving $\mathrm{P_w}$ then is as follows:

Algorithm WP(A, b, M, g, w, x°, ε).

Step 0 (Initialization).

$$\bar{x} = x^\circ,$$

$$\bar{w} = \min_i\{w_1, \ldots, w_m\},$$

$$r = \sqrt{\bar{w}/(1-\bar{w})},$$

$$R = \sqrt{(1-\bar{w})/\bar{w}},$$

$$F^* = +\infty.$$

Step 1 (Projective transformation to w-center).

$$\bar{s} = b - A\bar{x},$$

$$y = A^{\mathrm{T}}\bar{S}^{-1}w,$$

$$\tilde{A} = A - \bar{s}y^{\mathrm{T}}, \quad \tilde{b} = b - \bar{s}y^{\mathrm{T}}\bar{x},$$

$$(\tilde{c} = -y, \quad \tilde{U} = 1 - y^{\mathrm{T}}\bar{x}).$$

Step 2 (Optimization over inner ellipsoid). Solve the program:

$$\text{EP:} \quad \text{maximize} \quad -y^{\mathrm{T}}d$$

$$\text{subject to} \quad d^{\mathrm{T}}\tilde{A}^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}\tilde{A}d \leqslant r^2, \tag{5.14}$$

$$Md = 0.$$

The optimal solution is given by

$$\bar{d} = -\tilde{G}y/\sqrt{y^{\mathrm{T}}\tilde{G}y}, \tag{5.15}$$

where

$$\tilde{Q} = \tilde{A}^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}\tilde{A} \quad \text{and} \quad \tilde{G} = [\tilde{Q}^{-1} - \tilde{Q}^{-1}M^{\mathrm{T}}(M\tilde{Q}^{-1}M^{\mathrm{T}})^{-1}M\tilde{Q}^{-1}]. \tag{5.16}$$

If EP is unbounded from above, stop. $\mathrm{P_w}$ is unbounded.

Step 2a (Update upper bound on F^*).

$$\text{Set } \gamma = \gamma(\bar{x}) = (-y^{\mathrm{T}}\bar{d})/r^2. \tag{5.17}$$

$$\text{If } \gamma \geqslant 1/r^2, \text{ stop. Problem } \mathrm{P_w} \text{ is unbounded, and } \bar{d} \text{ is a ray of } \boldsymbol{X}. \tag{5.18}$$

$$\text{If } \gamma < 1, \quad F^* \leftarrow \min\left\{F^*, F_w(\bar{x}) + \gamma + \frac{\gamma^2}{2(1-\gamma)}\right\}. \tag{5.19}$$

$$\text{If } \gamma \leqslant \tfrac{1}{8}, \quad F^* \leftarrow \min\{F^*, F_w(\bar{x}) + (0.82)r^2\gamma^2\}. \tag{5.20}$$

Step 3 (Take step in the set $\boldsymbol{Z}$).

$$\alpha = 1 - 1/\sqrt{1+2\gamma},$$

$$z_{\mathrm{NEW}} = \bar{x} + \alpha\bar{d}.$$

Step 4 (Transform back to the set $\boldsymbol{X}$).

$$x_{\mathrm{NEW}} = \bar{x} + \frac{z_{\mathrm{NEW}} - \bar{x}}{1 + y^{\mathrm{T}}(z_{\mathrm{NEW}} - \bar{x})}.$$

Step 5 (Stopping criterion).

$$\text{Set } \bar{x} = x_{\mathrm{NEW}}. \quad \text{If } F_w(\bar{x}) \geqslant F^* - \varepsilon, \text{ stop.} \tag{5.21}$$

Otherwise, go to Step 1.

The data for the problem is the data (A, b, M, g) of the constraint set X, the vector w of positive weights that satisfy $e^{\mathrm{T}}w = 1$, an initial feasible solution x° of P_w, and an optimality tolerance $\varepsilon > 0$. We can assume without loss of generality that the constraint set X satisfies condition (A) of (3.10) by prior knowledge of the boundedness of X or by adding the null constraint $0^{\mathrm{T}}x \leq 1$ to the system (A, b). In Step 0, the value of $\bar{x}$ is initialized and the constants $\bar{w}$, r, and R of (2.8) are computed. In Step 1, the value of y of (4.1) is computed, and the constraint set data is transformed according to (3.4). In addition, we have from (5.3) and (5.5) that

$$\tilde{c} = -y \quad \text{and} \quad \tilde{U} = 1 - y^{\mathrm{T}}\bar{x}. \tag{5.22}$$

In Step 2, the inner ellipsoid program of (5.7) is solved via (5.8) for the transformed data. In Step 2a, the upper bound F^* is updated. The bounds given in (5.19) and (5.20) will be proven in Section 6. (The unboundedness criteria of (5.16) and (5.18) will be proven below in Lemma 5.3.) In Step 3, the stepsize α is computed according to (5.9) and (5.11). Note that the computation of γ from (5.9) is

$$\gamma = \frac{\tilde{c}^{\mathrm{T}}\bar{d}}{(\tilde{U} - \tilde{c}^{\mathrm{T}}\bar{x})r^2} = \frac{-y^{\mathrm{T}}\bar{d}}{r^2},$$

as is stated in (5.17). In Step 4, the new value of $z = z_{\mathrm{NEW}}$ (in the transformed set Z) is transformed back to the set X via the projective transformation $h(z) = h_{y,\bar{x}}(z)$ of (3.9). In Step 5, the optimality tolerance criterion is checked.

Lemma 5.1, Theorem 5.1 and Corollary 5.1 combine to yield the following:

Lemma 5.2 (performance of Algorithm WP). *At each iteration of Algorithm WP,*

$$F_w(x_{\mathrm{NEW}}) \geq F_w(\bar{x}) + r^2(1 + \gamma - \sqrt{1 + 2\gamma}),$$

where $\gamma = \gamma(\bar{x})$ *is defined in* (5.17). □

Remark 5.1 (use of line-search). Steps 3 and 4 can be augmented by a line-search of the objective function $F_w(x)$, without affecting the conclusion of Lemma 5.2. Because the projective transformation $g(\cdot)$ preserves directions from $\bar{x}$ one can perform the line-search in the space X directly. Specifically, one can replace the computation of α in Step 3 and all of Step 4 by finding a value $\delta \geq 0$ for which $F_w(\bar{x} + \delta\bar{d})$ is approximately maximized. As shown in Todd and Burrell [26], there will be only one local maximum of $F_w(\bar{x} + \delta\bar{d})$ for $\delta \geq 0$. The search could be started with $\delta = \alpha/(1 + \alpha y^{\mathrm{T}}\bar{d})$, where α is given in Step 3, which corresponds to the value of α in (5.11).

Remark 5.2 (interpretations of γ). The quantity γ in (5.9) is closely related to the length of the Newton step; see [10]. Let γ_{N} denote the norm of the Newton step for problem P_w at the point $\bar{x}$, using the norm defined by Hessian of the objective function at $\bar{x}$. Then $(\gamma/\gamma_{\mathrm{N}}) \to 1$ as $\bar{x} \to \hat{x}$, where $\hat{x}$ is the optimal solution to problem P_w; see [10].

Lemma 5.3 (detecting unboundedness in Algorithm WP).

(i) *If Algorithm* WP *stops via* (5.16), *then* P_w *is unbounded.*

(ii) *If Algorithm* WP *stops via* (5.18), *then* P_w *is unbounded.*

Proof. (i) If program EP of (5.14) has no solution, there exists a vector d for which $d^T\tilde{Q}d = 0$, $Md = 0$, and $-y^Td > 0$ (where $\tilde{Q}$ is defined in (5.15)). Thus $\tilde{A}d = 0$, i.e., $Ad = \bar{s}y^Td$. If X is bounded, then $d = 0$, contradicting $-y^Td > 0$.

(ii) In this case, from Lemma 5.1 and Theorem 5.1(i), program $\widetilde{\text{CP}}$ is unbounded from below, and so program CP is unbounded from below. □

However, Algorithm WP does not always detect unboundedness; see Remark 7.4.

Finally, we note that program CP of (5.1) can be reduced to the special case (2.6) of program P_w under the projective transformation

$$z = x/(U - c^Tx),$$

as is done in Bayer and Lagarias [4] for linear programming. Because algorithm WP is invariant under projective transformation (this is established by straightforward arithmetic), algorithm WP can be used to solve the more general program CP with all of the properties presented in this section (and in Sections 6 and 7 as well).

6. Linear and superlinear convergence of Algorithm WP

The purpose of this section is to establish the following four results regarding Algorithm WP for solving the w-center problem P_w.

Lemma 6.1 (optimal objective value bounds). *At Step* 2a *of Algorithm* WP:

(i) *If* $\gamma < 1$, P_w *has an optimal solution* $\hat{x}$, *and*

$$F_w(\hat{x}) \leq F_w(\bar{x}) + \gamma + \frac{\gamma^2}{2(1-\gamma)}.$$

(ii) *If* $\gamma \leq \frac{1}{8}$, *then* $F_w(\hat{x}) \leq F_w(\bar{x}) + (0.82)r^2\gamma^2$.

Lemma 6.1 validates the upper bounds computed in (5.19) and (5.20) of the algorithm.

Lemma 6.2 (local improvement). *At Step* 2a *of Algorithm* WP:

(i) *If* $\gamma \geq \frac{1}{8}$, $F_w(x_{\text{NEW}}) \geq F_w(\bar{x}) + (0.0069)r^2$.

(ii) *If* $\gamma \leq \frac{1}{8}$, *then* $F_w(x_{\text{NEW}}) \geq F_w(\bar{x}) + (0.44)r^2\gamma^2$.

Lemma 6.3 (linear convergence or fixed improvement). *At each iteration of Algorithm* WP, *at least one of the following is true*:

(i) $F_w(x_{\text{NEW}}) \geq F_w(\bar{x}) + (0.0069)r^2$.

(ii) $F_w(\hat{x}) - F_w(x_{\text{NEW}}) \leq (0.46)(F_w(\hat{x}) - F_w(\bar{x}))$, *where* $\hat{x}$ *is the w-center of* X.

Lemma 6.3 states that each iteration achieves either constant improvement (i) or linear convergence (ii) with a convergence upper bound constant of 0.46. The next theorem states that this upper bound constant will go to zero in the limit, thus establishing superlinear convergence.

Theorem 6.1. *If program* $\mathrm{P_w}$ *is bounded, then Algorithm* WP *exhibits superlinear convergence.*

The proofs of these results will make use of the following functions, defined below for convenience.

$$k(\gamma)=1+\gamma-\sqrt{1+2\gamma}, \quad \gamma\geq 0, \tag{6.1}$$

$$j(\theta)=k(\theta)/\theta^2, \quad \theta>0, \tag{6.2}$$

$$p(h)=\frac{h-\ln(1+h)}{h^2}, \tag{6.3}$$

$$q(h)=\tfrac{1}{2}(1+hp(h)-\sqrt{1+(hp(h))^2}), \quad h>0, \tag{6.4}$$

$$v(h)=p(h)-\frac{(q(h))^2}{2(1-q(h))}, \quad h>0, \tag{6.5}$$

$$m(h)=k(q(h)), \tag{6.6}$$

$$n(h)=j(q(h)). \tag{6.7}$$

Inequalities relating to these functions can be found in Propositions A.4–A.9 of the Appendix. We first will prove Lemma 6.1(i). The proof of Lemma 6.1(ii) is more involved.

Proof of Lemma 6.1(i). Under the projective transformation $g(x)=g_{y,\bar{x}}(x)$ where $y=A^{\mathrm{T}}\bar{S}^{-1}w$, $\bar{x}$ is the w-center of the system $\boldsymbol{Z}=(\tilde{A}, \tilde{b}, M, g)$ and problem $\mathrm{P_w}$ (2.6) is transformed, as in Lemma 5.1, to the program

$$\tilde{\mathrm{P}}_{\mathrm{w}}: \quad \underset{z,t}{\text{maximize}} \quad \tilde{F}_w(z)=-\ln(1+y^{\mathrm{T}}(z-\bar{x}))+\sum_{i=1}^{m} w_i \ln(\tilde{b}_i-\tilde{A}_i z)$$

$$\begin{aligned} \text{subject to} \quad & \tilde{A}z+t=\tilde{b},\\ & t>0,\\ & Mz=g,\\ & -y^{\mathrm{T}}z<1-y^{\mathrm{T}}\bar{x}. \end{aligned} \tag{6.8}$$

Because $\bar{x}$ is the w-center of $\boldsymbol{Z}$, then

$$\sum_{i=1}^{m} w_i \ln(\tilde{b}_i-\tilde{A}_i z)\leq \sum_{i=1}^{m} w_i \ln(\bar{s}_i)=\tilde{F}_w(\bar{x}), \tag{6.9}$$

for all $z \in \boldsymbol{Z}$. Also, because $\bar{x}+\bar{d}$ maximizes $-y^{\mathrm{T}}z$ over the ellipsoid E_{IN} of the $\boldsymbol{Z}$ polytope (defined in Theorem 2.1), then $\bar{x}+\bar{d}R/r$ maximizes $-y^{\mathrm{T}}z$ over the outer ellipsoid E_{OUT} of the $\boldsymbol{Z}$ polytope (also defined as in Theorem 2.1). Because $\boldsymbol{Z}\subset E_{\mathrm{OUT}}$, $-y^{\mathrm{T}}z \leq -y^{\mathrm{T}}(\bar{x}+\bar{d}R/r)$ for all $z\in\boldsymbol{Z}$. Put another way,

$$-y^{\mathrm{T}}(z-\bar{x}) \leq \gamma \quad \text{for all } z\in\boldsymbol{Z}. \tag{6.10}$$

This follows because $-y^{\mathrm{T}}\bar{d}R/r = -y^{\mathrm{T}}\bar{d}/r^2 = \gamma$. Therefore

$$\begin{aligned}\tilde{F}_w(z) &= -\ln(1+y^{\mathrm{T}}(z-\bar{x})) + \sum_{i=1}^{m} w_i \ln(\tilde{b}_i - \tilde{A}_i z) \\ &\leq -\ln(1+y^{\mathrm{T}}(z-\bar{x})) + \tilde{F}_w(\bar{x}) \quad \text{(from 6.9)} \\ &\leq \gamma + \frac{\gamma^2}{2(1-\gamma)} + \tilde{F}_w(\bar{x}) \quad \text{(from Proposition A.2).}\end{aligned}$$

Therefore, from the equivalence of $\tilde{\mathrm{P}}_{\mathrm{w}}$ and P_{w} under the projective transformation $g(x)=g_{y,\bar{x}}(x)$ and Lemma 5.1, $F_w(x)-F_w(\bar{x}) \leq \gamma+\gamma^2/(2(1-\gamma))$ for all $x\in \operatorname{int}\boldsymbol{X}$. □

The proof of Lemma 6.1(ii) will follow as a consequence of the following three lemmas.

Lemma 6.4. *Let $h>0$ be a given parameter. Suppose $\bar{x}$ is the w-center of $\boldsymbol{X}$, let $\bar{s}=b-A\bar{x}$, and suppose $\hat{x}\in\boldsymbol{X}$ satisfies:*

$$(\hat{x}-\bar{x})^{\mathrm{T}}A^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}A(\hat{x}-\bar{x}) = \beta^2.$$

Then

$$\sum_{i=1}^{m} w_i \ln(b_i - A_i\hat{x}) - \sum_{i=1}^{m} w_i \ln \bar{s}_i \leq \begin{cases} -p(h)\beta^2 & \text{if } \beta\leq hr, \\ -p(h)hr\beta & \text{if } \beta\geq hr, \end{cases}$$

where r is defined in (2.8) *and $p(h)$ is defined in* (6.3).

Proof. First observe that

$$\sum_{i=1}^{m} w_i \ln(b_i - A_i\hat{x}) - \sum_{i=1}^{m} w_i \ln \bar{s}_i = \sum_{i=1}^{m} w_i \ln(1+v_i),$$

where $v=\bar{S}^{-1}A(\bar{x}-\hat{x})$. Then note that $w^{\mathrm{T}}v = w^{\mathrm{T}}\bar{S}^{-1}A(\bar{x}-\hat{x}) = \bar{\pi}^{\mathrm{T}}M(\bar{x}-\hat{x})=0$, for some $\bar{\pi}\in\mathbb{R}^k$, from (2.7d). Also $v^{\mathrm{T}}Wv=\beta^2$. Therefore $(vr/\beta)^{\mathrm{T}}W(vr/\beta)=r^2$, and so from Proposition 2.2, $|v_ir/\beta|\leq 1$, i.e.,

$$|v_i| \leq \beta/r, \quad i=1,\ldots,m. \tag{6.11}$$

We now prove the two cases of the lemma.

Case 1 ($\beta\leq hr$). In this case $|v_i|\leq h$. From Proposition A.7, $\ln(1+v_i)\leq v_i - p(h)(v_i)^2$. Summing over i yields

$$\sum_{i=1}^{m} w_i \ln(1+v_i) \leq w^{\mathrm{T}}v - p(h)v^{\mathrm{T}}Wv = -p(h)\beta^2.$$

Case 2 ($\beta \geqslant hr$). In this case, from Proposition A.3 (with $a = (hr/\beta)$, $b = v_i$),

$$\sum_{i=1}^{m} w_i \ln(1+v_i) \leqslant \left(\frac{\beta}{hr}\right) \sum_{i=1}^{m} w_i \ln\left(1+\frac{rh}{\beta} v_i\right).$$

However, from (6.11), $(rh/\beta)v_i \leqslant h$, and so from Proposition A.7,

$$\sum_{i=1}^{m} w_i \ln(1+v_i) \leqslant \left(\frac{\beta}{hr}\right)\left(\frac{rh}{\beta} w^{\mathrm{T}} v - \frac{r^2h^2}{\beta^2} v^{\mathrm{T}} W v p(h)\right)$$

$$= -\frac{rh}{\beta} \beta^2 p(h) = -p(h) hr\beta. \qquad \square$$

Lemma 6.5. *Let $\bar{x}$ be the current iterate of Algorithm* WP *and let $\bar{s}$, y, $\tilde{b}$, $\tilde{A}$, $\tilde{Q}$, and γ be as defined in Steps* 1, 2, *and* 3. *Suppose $\hat{x}$ is the optimal solution to* $\mathrm{P_w}$, *and let $\hat{z} = g(\hat{x}) = g_{y,\bar{x}}(\hat{x})$. Suppose*

$$(\hat{z}-\bar{x})^{\mathrm{T}} \tilde{Q} (\hat{z}-\bar{x}) = \beta^2. \tag{6.12}$$

If $h > 0$ is a given parameter and $\gamma < 1$, then

$$F_w(\hat{x}) - F_w(\bar{x}) \leqslant \begin{cases} -p(h)\beta^2 + \beta r\gamma + \dfrac{\beta^2 r^2 \gamma^2}{2(1-\gamma)} & \text{if } \beta \leqslant hr, \\[2ex] -p(h) hr\beta + \beta r\gamma + \dfrac{\beta^2 r^2 \gamma^2}{2(1-\gamma)} & \text{if } \beta \geqslant hr. \end{cases}$$

Proof. Let $\tilde{\mathrm{P}}_{\mathrm{w}}$ be the projectively transformed equivalent program of $\mathrm{P_w}$, i.e., program (6.8). Then it suffices to show that $\tilde{F}_w(\hat{w}) - \tilde{F}_w(\bar{x})$ is less than or equal to the expressions in parentheses, by Lemma 5.1.

From Lemma 6.4,

$$\sum_{i=1}^{m} w_i \ln(\tilde{b}_i - \tilde{A}_i \hat{z}) - \sum_{i=1}^{m} w_i \ln(\bar{s}_i) \leqslant \begin{cases} -p(h)\beta^2 & \text{if } \beta \leqslant hr, \\ -p(h) hr\beta & \text{if } \beta \geqslant hr. \end{cases} \tag{6.13}$$

It thus remains to show that

$$-\ln(1 + y^{\mathrm{T}}(\hat{z}-\bar{x})) \leqslant \beta r\gamma + \frac{\beta^2 r^2 \gamma^2}{2(1-\gamma)}.$$

Let $\hat{d} = \hat{z} - \bar{x}$. Then from (6.12), $\pm \hat{d} r/\beta$ satisfies the constraints of EP (5.14), so that

$$\pm y^{\mathrm{T}} \hat{d} r/\beta \leqslant y^{\mathrm{T}} \bar{d} = \gamma r^2 \quad \text{(from 5.17)},$$

and consequently

$$|y^{\mathrm{T}} \hat{d}| \leqslant \gamma r\beta. \tag{6.14}$$

Furthermore, because of (6.12), from Theorem 2.1, we conclude that

$$\beta \leqslant R = 1/r, \quad \text{i.e., } r\beta \leqslant 1. \tag{6.15}$$

Finally, we obtain

$$\ln(1 + y^{\mathrm{T}}(\hat{z} - \bar{x})) = \ln(1 + y^{\mathrm{T}}\bar{d}) \geqslant \ln(1 - \gamma r\beta) \geqslant -\gamma r\beta - \frac{\gamma^2 r^2 \beta^2}{2(1-\gamma)},$$

from Proposition A.2, (6.14), and (6.15). □

Lemma 6.6. *Under the hypothesis of Lemma* 6.5,

$$\text{if } \gamma \leqslant q(h), \quad \text{then } \beta \leqslant hr,$$

where $q(h)$ *is defined in* (6.4).

Proof. Suppose $\beta > hr$. Then from Lemma 6.5,

$$F_w(\hat{x}) - F_w(\bar{x}) \leqslant f(\gamma, \beta), \tag{6.16}$$

where

$$f(\gamma, \beta) = -p(h)hr\beta + \beta r\gamma + \frac{\beta^2 r^2 \gamma^2}{2(1-\gamma)}.$$

Note that $f(\gamma, \beta)$ increases in γ for $\beta > 0$ and $0 \leqslant \gamma < 1$. Straightforward calculation reveals that $f(\gamma, \beta) = 0$ if

$$\gamma = \frac{1 + hp(h) - \sqrt{1 + (hp(h))^2 - 2hp(h) + 2hp(h)\beta r}}{2 - r\beta}$$
$$> \frac{1 + hp(h) - \sqrt{1 + (hp(h))^2}}{2} = q(h),$$

because $0 \leqslant r\beta \leqslant 1$ from (6.15). Thus if $\gamma \leqslant q(h), f(\gamma, \beta) < 0$, contradicting the optimality of $\hat{x}$ in (6.16). Therefore if $\gamma \leqslant q(h)$, $\beta \leqslant hr$. □

Proof of Lemma 6.1(ii). We will actually prove a stronger result, namely:

$$\text{If } 0 < h \leqslant 1, \text{ and } \gamma \leqslant q(h), \quad \text{then } F_w(\hat{x}) - F_w(\bar{x}) \leqslant \left(\frac{1}{4v(h)}\right)\gamma^2 r^2, \tag{6.17}$$

where $v(h)$ is defined in (6.5). (ii) will follow by substituting $h = 0.93$. Then $q(h) \geqslant \frac{1}{8}$, and $(1/4v(h)) \leqslant 0.82$. To prove (6.17), observe that if $\gamma \leqslant q(h)$, from Lemma 6.6,

$\beta \leqslant hr$, and so from Lemma 6.5,

$$F_w(\hat{x}) - F_w(\bar{x}) \leqslant -p(h)\beta^2 + \beta r\gamma + \frac{\beta^2 r^2 \gamma^2}{2(1-\gamma)}$$

$$\leqslant -p(h)\beta^2 + \beta r\gamma + \frac{\beta^2\gamma^2}{2(1-\gamma)}$$

$$= -\left(p(h) - \frac{\gamma^2}{2(1-\gamma)}\right)\beta^2 + \beta r\gamma. \tag{6.18}$$

However, $(p(h) - \gamma^2/(2(1-\gamma))) \geqslant v(h)$ because $\gamma \leqslant q(h)$, and so from Proposition A.9, $(p(h) - \gamma^2/(2(1-\gamma))) \geqslant v(h) > 0$ for $h \leqslant 1$, so that the bound of (6.18) is a concave quadratic in β. The maximum possible value of the bound is then given by $\beta = r\gamma/(2p(h) - \gamma^2/(1-\gamma))$, which yields from (6.18),

$$F_w(\hat{x}) - F_w(\bar{x}) \leqslant \frac{r^2\gamma^2}{4p(h) - 2\gamma^2/(1-\gamma)} \leqslant \frac{r^2\gamma^2}{4v(h)}. \qquad \square$$

Proof of Lemma 6.2. (i) We will actually prove a stronger result, namely:

$$\text{If } 0 < h \leqslant 1, \text{ and } \gamma \geqslant q(h), \text{ then } F_w(x_{\text{NEW}}) - F_w(\bar{x}) \geqslant m(h)r^2, \tag{6.19}$$

where $m(h)$ is defined in (6.6). (i) will follow by setting $h = 0.923$. Then $q(h) \leqslant \frac{1}{8}$, and $m(h) \geqslant 0.0069$. To prove (6.19), observe from Lemma 5.2 and Theorem 5.1 that

$$F_w(x_{\text{NEW}}) \geqslant F_w(\bar{x}) + r^2(1 + \gamma - \sqrt{1+2\gamma})$$

$$= F_w(\bar{x}) + r^2 k(\gamma) \quad (\text{see } (6.1))$$

$$\geqslant F_w(\bar{x}) + r^2 k(q(h)) = F_w(\bar{x}) + r^2 m(h), \tag{6.20}$$

from Proposition A.8.

(ii) We will prove a stronger result, namely:

$$\text{If } 0 < h \leqslant 1, \text{ and } \gamma \leqslant q(h), \text{ then } F_w(x_{\text{NEW}}) - F_w(\bar{x}) \geqslant n(h)r^2\gamma^2, \tag{6.21}$$

where $n(h)$ is defined in (6.7). (ii) will follow by setting $h = 0.929$. Then $q(h) \geqslant \frac{1}{8}$ and $n(h) \geqslant 0.44$. To prove (6.21), observe from (6.20) that

$$F_w(x_{\text{NEW}}) \geqslant F_w(\bar{x}) + r^2 k(\gamma)$$

$$\geqslant F_w(\bar{x}) + r^2 j(\theta)\gamma^2 \quad \text{for } 0 \leqslant \gamma \leqslant \theta \quad (\text{from Proposition A.5})$$

$$= F_w(\bar{x}) + r^2 j(q(h))\gamma^2 \quad (\text{substituting } q(h) = \theta).$$

$$= F_w(\bar{x}) + n(h)r^2\gamma^2 \quad (\text{from } (6.7)). \qquad \square$$

Proof of Lemma 6.3. Let $h = 0.929$. (i) follows from (6.19) if $\gamma \geq q(h)$.
Now suppose $\gamma \leq q(h)$. Then from (6.17) and (6.21),

$$\frac{F_w(\hat{x}) - F_w(x_{\text{NEW}})}{F_w(\hat{x}) - F_w(\bar{x})} = 1 - \frac{F_w(x_{\text{NEW}}) - F_w(\bar{x})}{F_w(\hat{x}) - F_w(\bar{x})} \leq 1 - \frac{n(h) r^2 \gamma^2}{(1/4v(h)) r^2 \gamma^2}$$

$$= 1 - 4n(h)v(h) \leq 0.46. \quad \square \tag{6.22}$$

Proof of Theorem 6.1. It suffices to prove that as $h \to 0$, the convergence constant of (6.22) goes to zero. This constant is

$$1 - 4v(h)n(h) = 1 - 4\left(p(h) - \frac{q(h)^2}{(1 - q(h))}\right)(j(q(h))).$$

From Propositions A.4, A.6, and A.8, $p(h) \to \frac{1}{2}$, $q(h) \to 0$, and $j(q(h)) \to \frac{1}{2}$, as $h \to 0$. Thus

$$1 - 4v(h)n(h) \to 1 - 4(\tfrac{1}{2})(\tfrac{1}{2}) = 0 \quad \text{as } h \to 0. \quad \square$$

Remark 6.1 (monotonic decrease of γ). One natural question to ask regarding Algorithm WP is whether the values of γ generated at each iteration are monotonically decreasing. We have:

Proposition 6.1. *Suppose γ_1 and γ_2 are two successive values of γ generated by Algorithm* WP. *Then if $\gamma_1 \leq \frac{1}{8}$, $\gamma_2 \leq (0.92)\gamma_1$.*

Proof. Let x_1 and x_2 be the successive iterates of Algorithm WP that generate the values of $\gamma = \gamma_1$ and $\gamma = \gamma_2$, respectively, and let x_3 be the iterate value of x after x_2. Let $h = 0.93$. Then $\gamma_1 \leq q(h)$. Suppose γ_2 does not satisfy $\gamma_2 \leq q(h)$. Then from (6.17), (6.19), and (6.21),

$$F_w(x^3) \leq F_w(\hat{x}) \leq F_w(x^1) + \left(\frac{1}{4v(h)}\right)\gamma_1^2 r^2, \tag{6.23}$$

$$\begin{aligned} &F_w(x^3) \geq F_w(x^2) \geq F_w(x^1) + n(h)\gamma_1^2 r^2, \\ &F_w(\hat{x}) \geq F_w(x^3) \geq F_w(x^2) + m(h) r^2. \end{aligned} \tag{6.24}$$

Combining the above inequalities yields

$$\left(\frac{1}{4v(h)}\right)\gamma_1^2 \geq m(h) + n(h)\gamma_1^2,$$

i.e.,

$$\gamma_1 \geq \sqrt{\frac{m(h)}{4v(h) - n(h)}}.$$

However, $\gamma_1 \leq q(h)$, which is a contradiction at $h = 0.93$. Thus $\gamma_2 \leq q(h)$. This being the case, from (6.21) we obtain

$$F_w(x^3) - F_w(x^2) \geq n(h)\gamma_2^2 \tag{6.25}$$

which in combination with (6.23) and (6.24) yields

$$\gamma_2 \leq \left(\sqrt{\frac{1/(4v(h)) - n(h)}{n(h)}}\right)\gamma_1 \leq 0.92\gamma_1. \qquad \square$$

7. The improving direction is the Newton direction

In this section, we show that the direction $\bar{d}$ of Step 2 of Algorithm WP is a positively scaled projected Newton direction. As a byproduct of this result, the computation of $\bar{d}$ in Step 2 can be carried out without solving equations involving the matrix $\tilde{Q} = \tilde{A}^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}\tilde{A}$, which will typically be extremely dense. Vaidya's algorithm for the center problem [28] corresponds to computing the Newton direction and performing an inexact line-search. Algorithm WP specializes to Vaidya's algorithm when the algorithm is augmented with a line-search; see Remark 5.1. Furthermore, this establishes that Vaidya's algorithm then will exhibit superlinear convergence.

Let $\bar{x}$ be the current iterate of Algorithm WP, let $\bar{s} = b - A\bar{x}$, $y = A^{\mathrm{T}}\bar{S}^{-1}w$ and $\tilde{A} = A - \bar{s}y^{\mathrm{T}}$ as in Step 1 of Algorithm WP, and let $Q = A^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}A$, and $\tilde{Q} = \tilde{A}^{\mathrm{T}}\bar{S}^{-1}W\bar{S}^{-1}\tilde{A}$. By assumption, A has full rank, so that Q is nonsingular and positive definite. Let $F_w(x)$ be the weighted logarithmic barrier objective function of P_{w} given in (2.6). Then the gradient of $F_w(\cdot)$ at $\bar{x}$ is given by $-y$, i.e., $\nabla F_w(\bar{x}) = -y$, and the Hessian of $F_w(\cdot)$ at $\bar{x}$ is given by $-Q$, i.e., $\nabla^2 F_w(\bar{x}) = -Q$. Thus the projected Newton direction d_{N} at $\bar{x}$ is the optimal solution to

$$\begin{aligned} &\text{maximize} \quad -y^{\mathrm{T}}d - \tfrac{1}{2}d^{\mathrm{T}}Qd \\ &\text{subject to} \quad Md = 0, \end{aligned} \tag{7.1}$$

and the Newton direction d_{N} together with Lagrange multipliers π_{N} is the unique solution to

$$\begin{aligned} &Qd_{\mathrm{N}} - M^{\mathrm{T}}\pi_{\mathrm{N}} = -y, \\ &Md_{\mathrm{N}} = 0. \end{aligned} \tag{7.2}$$

Because Q has rank n and M has rank k, we can write the solution to (7.2) as

$$d_{\mathrm{N}} = -Q^{-1}y + Q^{-1}M^{\mathrm{T}}\pi_{\mathrm{N}}, \tag{7.3}$$

where

$$\pi_{\mathrm{N}} = (MQ^{-1}M^{\mathrm{T}})^{-1}MQ^{-1}y.$$

Theorem 7.1 (positive scaled Newton direction). *Let d_N be the Newton direction given by* (7.3), *and let* $\tilde{Q} = \tilde{A}^T \bar{S}^{-1} W \bar{S}^{-1} \tilde{A}$. *Then $\tilde{Q}$ is positive semi-definite and*:

(i) *If $d_N^T \tilde{Q} d_N > 0$, then d_N is the direction of Step* 2 *of Algorithm* WP.

(ii) *If $d_N^T \tilde{Q} d_N = 0$, then the direction d_N gives a nontrivial ray entirely in* $\boldsymbol{X}$, *problem* EP *of Step* 2 *of Algorithm* 2 *is unbounded from above, and* P_w *is unbounded from above.*

Proof. Consider the scaled vector

$$\bar{d} = d_N r / \sqrt{d_N^T \tilde{Q} d_N}. \tag{7.4}$$

(i) Let π_N be as given in (7.3), $\bar{\pi} = \pi_N / (1 + y^T d_N)$, and

$$\bar{\beta} = \sqrt{d_N^T \tilde{Q} d_N} / (2r(1 + y^T d_N)).$$

Then $\bar{d}, \bar{\pi}, \bar{\beta}$ satisfy the K-K-T conditions of program EP, namely $\bar{d}^T \tilde{Q} \bar{d} = r^2$, $M\bar{d} = 0$, and $-y = 2\bar{\beta}\tilde{Q}\bar{d} - M^T \bar{\pi}$, $\bar{\beta} > 0$, so long as $1 + y^T d_N > 0$. It thus remains to show that $1 + y^T d_N > 0$. Note first that $\tilde{Q} = Q - yy^T$, where $Q = A^T \bar{S}^{-1} W \bar{S}^{-1} A$. By hypothesis, we have

$$\begin{aligned} 0 < d_N^T \tilde{Q} d_N &= d_N^T (Q - yy^T) d_N \\ &= d_N^T Q d_N - (y^T d_N)^2 = -y^T d_N - (y^T d_N)^2, \end{aligned} \tag{7.5}$$

which implies $y^T d_N > -1$, i.e., $1 + y^T d_N > 0$.

(ii) Suppose $d_N^T \tilde{Q} d_N = 0$. In view of (7.5), we have $y^T d_N = -1$, and $d_N^T \tilde{Q} d_N = 0$, and $M d_N = 0$. Thus program EP is unbounded, and as in the proof of Lemma 5.3, d_N gives a nontrivial ray of $\boldsymbol{X}$. □

Remark 7.1 (simplified computation of $\bar{d}$). The formula (7.4) is better than (5.15) to compute $\bar{d}$ because it avoids inverting the possibly very dense matrix $\tilde{Q}$.

Remark 7.2 (relation of Algorithm WP to Vaidya's algorithm). Theorem 7.1 shows that $\bar{d}$ is just a positive scale of the Newton direction d_N. Suppose Algorithm WP is implemented with a line-search replacing Steps 3 and 4. Then because the projective transformations $g(x)$ and $h(z)$ given by (3.8) and (3.9) preserve directions from $\bar{x}$, the algorithm's direction in the space $\boldsymbol{X}$ will be d_N. Therefore, when using a line-search, the algorithm is just searching in the Newton direction. This is precisely Vaidya's algorithm [28], when all weights w_i are identical. And because the complexity analysis of Sections 5 and 6 carries through with or without a line-search, we see that Vaidya's algorithm exhibits superlinear convergence.

Remark 7.3 (an extension of a theorem of Bayer and Lagarias). In [4], Bayer and Lagarias have shown the following structural equivalence between Karmarkar's algorithm for linear programming and Newton's method: First one can projectively transform the problem of minimizing Karmarkar's potential function over a polyhedron $\boldsymbol{X}$ to finding the (unbounded) center of an unbounded polyhedron $\boldsymbol{Z}$, where $\boldsymbol{Z}$ is the image of $\boldsymbol{X}$ under a projective transformation that sends the set of optimal solutions to the linear program to the hyperplane at infinity. Then the image of Karmarkar's algorithm (with a line-search) in the space Z corresponds to performing a line-search in the Newton direction for the center problem in the transformed space Z. Theorem 7.1 generalizes this result. It states that if one is trying to find the center of any polyhedron X (bounded or not), then the direction generated at any iteration of the projective transformation method (i.e., Algorithm WP) is a positive scale of the Newton direction for the barrier function (2.6). Thus, if one determines step-lengths by a line-search of the objective function, then the projective transformation method corresponds to Newton's method with a line-search.

Another important relationship between directions generated by projective transformation methods and Newton's method can be found in Gill et al. [13].

Remark 7.4 (detecting unboundedness in Algorithm WP). Algorithm WP will not always detect unboundedness via (5.16) or (5.18). This is shown in an example of Section 4 of Bayer and Lagarias [4]. In that example, $X=\{x\in\mathbb{R}^2 \mid x_1\geq -1,\ x_1\leq -1,\ x_2\geq 0,\}$, $w=(\frac{1}{3},\frac{1}{3},\frac{1}{3})$, and the starting point of the Algorithm WP is $x^\circ=(\frac{1}{3},\frac{2}{3})$. They show that Newton's method (with a line-search) never produces a ray of X. As a consequence of Theorem 7.1, Algorithm WP (with a line-search) will never detect unboundedness for this example.

Finally, we point out that the objective function $F_w(x)$ of program P_w is a self-concordant function in the notation of Nesterov and Nemerovsky [20], who present a general analysis of Newton's method in this context; see in particular Theorem 1.2 of [9].

Appendix: Inequalities related to logarithms

Proposition A.1. $\ln(1+a)\leq a$.

Proof. Follows from the concavity of the logarithm function. □

Proposition A.2. *If* $|\alpha|\leq\delta<1$, *then* $\ln(1+\alpha)\geq\alpha-\alpha^2/(2(1-\delta))$.

Proof. See, e.g., Todd and Ye [27]. □

Proposition A.3. *If* $0<a\leq 1$ *and* $|b|<1$, *then* $\ln(1+b)\leq(1/a)\ln(1+ab)$.

Proof. $\ln(1+ab)=\ln(a(1+b)+(1-a)(1))\geq a\ln(1+b)+(1-a)\ln(1)=a\ln(1+b)$, where the inequality follows from the concavity of the logarithm function. □

Consider the functions $k(\gamma), j(\theta), p(h), q(h), v(h), m(h)$, and $n(h)$ defined in (6.1)-(6.7).

Proposition A.4. (i) $j(\theta)$ *is decreasing in* θ. (ii) $\lim_{\theta\to 0} j(\theta)=\frac{1}{2}$. □

Proposition A.5. $k(\gamma)\geqslant j(\theta)\gamma^2$ *for* $0\leqslant\gamma\leqslant\theta$,

Proof. Follows from Proposition A.4(i). □

Proposition A.6. (i) $p(h)$ *is decreasing in* h. (ii) $\lim_{h\to 0} p(h)=\frac{1}{2}$. □

Proposition A.7. $\ln(1+x)\leqslant x-p(h)x^2$ *for* $-1<x\leqslant h$.

Proof. Follows from Proposition A.6(i). □

Proposition A.8. (i) $q(h)$ *is increasing in* h. (ii) $\lim_{h\to 0} q(h)=0$. (iii) $0<q(h)<0.30$ *for all* $h>0$. (iv) $q(h)$ *is a concave function.* □

Proposition A.9. (i) $v(h)$ *is decreasing in* h. (ii) $v(h)>0$ *for* $h\leqslant 1$.

Proof. (i) Follows from Proposition A.6(i), A.8(i), and A.8(iii). Assertion (ii) follows from (i) and direct substitution. □

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