Online Companion for
Robust Controls for Network Revenue Management

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Proof of Proposition 1.

With a slight abuse of notation, let us denote by $\mathcal{D}_{\mathbf{d}}$ the set of multivariate stochastic processes such that the aggregate demand equals $\mathbf{d}$. Accordingly, Problem (5) can be decomposed as follows:

$$\begin{aligned}\rho(\mathbf{y}) &= \max_{\mathbf{d}\in\mathcal{P}} \max_{\mathbf{D}\in\mathcal{D}_{\mathbf{d}}} \left\{ \max_{\mathbf{z}\in\mathcal{F}} R(\mathbf{z},\mathbf{D}) - R(\mathbf{y},\mathbf{D}) \right\} \\ &= \max_{\mathbf{d}\in\mathcal{P}} \left\{ \max_{\mathbf{z}\in\mathcal{F}} \max_{\mathbf{D}\in\mathcal{D}_{\mathbf{d}}} R(\mathbf{z},\mathbf{D}) + \max_{\mathbf{D}\in\mathcal{D}_{\mathbf{d}}} -R(\mathbf{y},\mathbf{D}) \right\}.\end{aligned}$$

Let ξ_j be the realized sales under a booking policy $\mathbf{z}$ when the demand is equal to $\mathbf{d}$. Because the perfect hindsight booking limits $\mathbf{z}$ are chosen after observing the demand realization, one can assume without loss of generality that they are partitioned, i.e., $\xi_j \leq z_j$ for $j = 1, ..., N$. Hence, $\mathcal{F} = \{\mathbf{z} \geq \mathbf{0}\}$. Similar to (9), ξ_j must satisfy:

$$\xi_j = \min \left\{ d_j, \min_{k=1,...,K} \frac{1}{a_{kj}} \left\{ c_k - \sum_{i=1,i\neq j}^{N} a_{ki}\xi_i \right\}, z_j \right\} \quad j = 1, ..., N. \tag{A-1}$$

For any $\mathbf{d} \in \mathcal{P}$, different demand arrival processes $\mathbf{D} \in \mathcal{D}_{\mathbf{d}}$ are associated with different sales $\boldsymbol{\xi}$, but the sales will always satisfy (A-1). Let us denote by $\boldsymbol{\xi}(\mathbf{D})$ the realized sales under the demand process $\mathbf{D}$. Hence, for any $\mathbf{d} \in \mathcal{P}$,

$$\max_{\mathbf{D}\in\mathcal{D}_{\mathbf{d}}} R(\mathbf{z},\mathbf{D}) = R(\mathbf{z},\mathbf{D}^*) = \mathbf{r}'\boldsymbol{\xi}(\mathbf{D}^*) \leq \max_{\boldsymbol{\xi}:(A-1)\text{holds}} \mathbf{r}'\boldsymbol{\xi}.$$

On the other hand, for any sales vector $\boldsymbol{\xi}$, there always exists a demand arrival process that gives rise to those sales. Let us denote by $\mathcal{D}_{\mathbf{d}}(\boldsymbol{\xi})$ the set of demand processes that lead to the sales $\boldsymbol{\xi}$. Hence, for any $\mathbf{d} \in \mathcal{P}$,

$$\begin{aligned}\max_{\boldsymbol{\xi}:(A-1)\text{holds}} \mathbf{r}'\boldsymbol{\xi} = \mathbf{r}'\boldsymbol{\xi}^* &= R(\mathbf{z},\mathbf{D}) \quad \forall \mathbf{D} \in \mathcal{D}_{\mathbf{d}}(\boldsymbol{\xi}^*) \\ &\leq \max_{\mathbf{D}\in\mathcal{D}_{\mathbf{d}}} R(\mathbf{z},\mathbf{D}).\end{aligned}$$

Combining these two results, we find that, for any $\mathbf{d} \in \mathcal{P}$,

$$\max_{\boldsymbol{\xi}:(A-1)\text{holds}} \mathbf{r}'\boldsymbol{\xi} = \max_{\mathbf{D}\in\mathcal{D}_{\mathbf{d}}} R(\mathbf{z}, \mathbf{D}).$$

Similarly, one can show that, for any $\mathbf{d} \in \mathcal{P}$,

$$\max_{\mathbf{x}:(9)\text{holds}} -\mathbf{r}'\mathbf{x} = \max_{\mathbf{D}\in\mathcal{D}_{\mathbf{d}}} -R(\mathbf{y}, \mathbf{D}).$$

As a result,

$$\rho(\mathbf{y}) \quad = \quad \max_{\mathbf{d}\in\mathcal{P}} \left\{ \max_{\mathbf{z}\geq\mathbf{0}} \max_{\boldsymbol{\xi}:(A-1)\text{holds}} \mathbf{r}'\boldsymbol{\xi} + \max_{\mathbf{x}:(9)\text{holds}} -\mathbf{r}'\mathbf{x} \right\}.$$

On the one hand,

$$\begin{aligned} \max_{\mathbf{z}\geq\mathbf{0}} \max_{\boldsymbol{\xi}:(A-1)\text{holds}} \mathbf{r}'\boldsymbol{\xi} = \quad & \max\nolimits_{\boldsymbol{\xi},\mathbf{z}} \mathbf{r}'\boldsymbol{\xi} \\ s.t. \quad & \mathbf{A}\boldsymbol{\xi} \leq \mathbf{c} \\ & \mathbf{0} \leq \boldsymbol{\xi} \leq \mathbf{d} \\ & \boldsymbol{\xi} \leq \mathbf{z} \\ = \quad & \max\nolimits_{\mathbf{z}} \mathbf{r}'\mathbf{z} \\ s.t. \quad & \mathbf{A}\mathbf{z} \leq \mathbf{c} \\ & \mathbf{0} \leq \mathbf{z} \leq \mathbf{d}, \end{aligned}$$

in which the first equality follows because $r_j > 0$ for all $j = 1, ..., N$, and the second equality is obtained by observing that, if $(\boldsymbol{\xi}^*, \mathbf{z}^*)$ is an optimal solution, then $(\boldsymbol{\xi}^*, \boldsymbol{\xi}^*)$ is also optimal.

On the other hand,

$$\begin{aligned} \max_{\mathbf{x}:(9)\text{holds}} -\mathbf{r}'\mathbf{x} = \quad & \max\nolimits_{\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} -\mathbf{r}'\mathbf{x} & \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{c} & \\ & \textstyle\sum_{j\in S} x_j \leq y_S & S \in \mathcal{S} \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{d} & \\ & \mathbf{d} \leq \mathbf{x} + M(\mathbf{1} - \boldsymbol{\alpha}) & \\ & \textstyle\sum_{j\in S} x_j \geq \beta_S y_S & S \in \mathcal{S} \\ & \mathbf{a}_k'\mathbf{x} \geq c_k\gamma_k & k = 1, ..., K \\ & \textstyle\sum_{k=1:a_{kj}>0}^{K} \gamma_k + \alpha_j + \sum_{S:j\in S} \beta_S \geq 1 & j = 1, ..., N \\ & \boldsymbol{\alpha} \in \{0,1\}^N,\ \boldsymbol{\beta} \in \{0,1\}^{|\mathcal{S}|},\ \boldsymbol{\gamma} \in \{0,1\}^K. & \end{aligned} \tag{A-2}$$

where $M \geq \{\max_j d_j : \mathbf{d} \in \mathcal{P}\}$. The constraints involving the binary variables $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ ensure that (9) is satisfied. Because of the forcing constraint $d_j \leq x_j + M(1-\alpha_j)$, $\alpha_j = 1$ only if $x_j = d_j$. Similarly, $\beta_S = 1$ only if $\sum_{j\in S} x_j = y_S$, and $\gamma_k = 1$ only if $\mathbf{a}_k'\mathbf{x} = c_k$. The joint constraint $\sum_{k=1:a_{kj}>0}^{K} \gamma_k + \alpha_j + \sum_{S:j\in S} \beta_S \geq 1$ guarantees that at least one of these three scenarios occurs, for every j, so as to satisfy (9). Combining these results gives rise to the desired result.

Similarly, Problem (3) can be decomposed as follows:

$$\begin{aligned} \varphi(\mathbf{y}) &= -\max_{\mathbf{d}\in\mathcal{P}} \max_{\mathbf{D}\in\mathcal{D}_{\mathbf{d}}} -R(\mathbf{y},\mathbf{D}) \\ &= -\max_{\mathbf{d}\in\mathcal{P}} \max_{\mathbf{x}:(9)\text{holds}} -\mathbf{r}'\mathbf{x}. \end{aligned}$$

Using (A-2) completes the proof. □

Proof of Proposition 2.

(a) According to Proposition 1, evaluating the maximum regret on a single-resource RM problem with capacity c, with partitioned booking limits, i.e., $x_j \leq y_j$ for all $j = 1,...,N$, under interval uncertainty $\mathcal{P} = \{\mathbf{d} \in [\mathbf{0},\mathbf{u}]\}$ can be formulated as follows:

$$\begin{aligned} \max_{\mathbf{z},\mathbf{x},\mathbf{d}} \quad & \mathbf{r}'\mathbf{z} - \mathbf{r}'\mathbf{x} \\ s.t. \quad & \mathbf{0} \leq \mathbf{d} \leq \mathbf{u} \\ & \mathbf{1}'\mathbf{z} \leq c \\ & \mathbf{0} \leq \mathbf{z} \leq \mathbf{d} \\ & x_j = \min\{d_j, y_j, c - \textstyle\sum_{i=1,i\neq j}^{N} x_i\} \quad j = 1,...,N. \end{aligned} \tag{A-3}$$

Because capacity is partitioned into N buckets, $\sum_{j=1}^{N} y_j = c$. Accordingly, the last constraint can be replaced with $x_j = \min\{d_j, y_j\}$, simplifying Problem (A-3) to:

$$\begin{aligned} \max_{\mathbf{z},\mathbf{d}} \quad & \textstyle\sum_{j=1}^{N} r_j\,(z_j - \min\{y_j, d_j\}) \\ s.t. \quad & \mathbf{0} \leq \mathbf{d} \leq \mathbf{u} \\ & \mathbf{1}'\mathbf{z} \leq c \\ & \mathbf{0} \leq \mathbf{z} \leq \mathbf{d}. \end{aligned}$$

Because the objective function is decreasing with $\mathbf{d}$, $(\mathbf{z}^*, \mathbf{z}^*)$ is optimal whenever $(\mathbf{z}^*, \mathbf{d}^*)$ is optimal. Therefore, Problem (A-3) can be further simplified to:

$$\max_{\mathbf{z}} \quad \textstyle\sum_{j=1}^{N} r_j\,(z_j - y_j)^+$$

$$\begin{aligned} s.t. \quad & \mathbf{0} \leq \mathbf{z} \leq \mathbf{u} \\ & \mathbf{1}'\mathbf{z} \leq c. \end{aligned}$$

Without loss of optimality, one can assume that the optimal solution $\mathbf{z}^*$ is such that either $z_j^* > y_j$ or $z_j^* = 0$ for all j. If there exists some j for which $0 < z_j^* \leq y_j$, then taking $z_j^* = 0$ maintains feasibility of the solution without decreasing the value of the objective function. Accordingly, we can assume that, whenever $z_j > 0$, a fixed charge $r_j y_j$ must be incurred, for any j. For any j, let δ_j be a binary variable equal to one when $z_j > 0$ and to zero otherwise. Accordingly, Problem (A-3) is equivalent to:

$$\begin{aligned} \max_{\mathbf{z},\boldsymbol{\delta}} \quad & \textstyle\sum_{j=1}^{N} r_j \left(z_j - y_j \delta_j \right) \\ s.t. \quad & \mathbf{1}'\mathbf{z} \leq c \\ & 0 \leq z_j \leq u_j \delta_j \qquad j = 1, ..., N \\ & \boldsymbol{\delta} \in \{\mathbf{0}, \mathbf{1}\}^N. \end{aligned} \tag{A-4}$$

Therefore, for any problem instance $(c, \mathbf{u}, \mathbf{y})$ and any $b \in \mathbb{R}$, $\rho(\mathbf{y}) \geq b$ if and only if the optimal value of (A-4) is greater than or equal to b. Moreover, if $(\mathbf{z}^*, \boldsymbol{\delta}^*)$ solves (A-4), then one can construct a solution $(\mathbf{z}, \mathbf{x}, \mathbf{d}) = (\mathbf{z}^*, \min\{\mathbf{z}^*, \mathbf{y}\}, \mathbf{z}^*)$ that solves (A-3) and conversely, if $(\mathbf{z}^*, \mathbf{x}^*, \mathbf{d}^*)$ solves (A-3), then one can construct a solution $(\mathbf{z}, \boldsymbol{\delta})$ that solves (A-4) with $z_j = z_j^*$ and $\delta_j = 1$ if $z_j^* > y_j$, and $z_j = 0$ and $\delta_j = 0$ otherwise. Hence, Problems (A-3) and (A-4) are equivalent.

We next show that the following KNAPSACK problem, which is known to be NP-complete (Garey and Johnson 1979, [MP9]), is reducible to (A-4):

- INSTANCE: Finite set U, for each $j \in U$, a weight $w_j \in \mathbb{Z}^+$ and a value $v_j \in \mathbb{Z}^+$, and positive integers κ and b.
- QUESTION: Is there a subset $U' \subseteq U$ such that $\sum_{j \in U'} w_j \leq \kappa$ and $\sum_{j \in U'} v_j \geq b$?

Given any instance of KNAPSACK, we define the following instance of the maximum regret RM problem (A-4): $\{1, ..., N\} = U$, $K = 1$, $c = \kappa$ and, for any $j \in U$, $u_j = w_j$, $y_j = w_j - 1$, $r_j = v_j$. Hence, the maximum regret problem can be formulated as follows:

$$\begin{aligned} \max_{\mathbf{z},\boldsymbol{\delta}} \quad & \textstyle\sum_{j \in U} v_j \left(z_j - (w_j - 1)\delta_j \right) \\ s.t. \quad & \mathbf{1}'\mathbf{z} \leq \kappa \end{aligned}$$

$$0 \leq z_j \leq w_j \delta_j \qquad j \in U \tag{A-5}$$
$$\boldsymbol{\delta} \in \{\mathbf{0}, \mathbf{1}\}^{|U|}.$$

Because (A-5) is a convex maximization problem, there exists an optimal solution that is an extreme point of the feasible set. That is, there is at most one product, say product i, for which $0 < z_i < w_i$. Because all data is integer, z_i can only take on integer values in an optimal solution, i.e., $1 \leq z_i \leq w_i - 1$. However, the objective can be increased or remains unchanged by setting $z_i = 0$ because the fixed charge equals $v_i(w_i - 1)$. Therefore, there exists an optimal solution with $z_j = 0$ or $z_j = w_j$ for every $j \in U$.

Let $(\mathbf{z}^*, \boldsymbol{\delta}^*)$ be an optimal solution to (A-5), with optimal value greater than or equal to b. Define $U' = \{j : \delta_j^* = 1\}$. Then, $\kappa \geq \sum_{j \in U} z_j^* = \sum_{j \in U'} w_j$ and $b \leq \sum_{j \in U} v_j \left(z_j^* - (w_j - 1)\delta_j^*\right) = \sum_{j \in U'} v_j \left(w_j - (w_j - 1)\right) = \sum_{j \in U'} v_j$; therefore, the answer to KNAPSACK is positive. Conversely, suppose the answer to KNAPSACK is positive. Setting $\delta_j = 1$ and $z_j = w_j$ if $j \in U'$, and $\delta_j = 0$ and $z_j = 0$ if $j \in U \setminus U'$ gives rise to a feasible solution, with an objective value greater than or equal to b. As a result, Problem (A-3) is NP-Hard.

(b) According to Proposition 1, evaluating the minimum revenue on a single-resource RM problem with capacity c, with partitioned booking limits, i.e., $x_j \leq y_j$ for all $j = 1, ..., N$, under interval uncertainty $\mathcal{P} = \{\mathbf{d} \in [\mathbf{0}, \mathbf{u}]\}$ and with a lower bound on the total demand $\sum_{j=1}^N d_j \geq L$ can be formulated as follows:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{d}} \quad & \mathbf{r}'\mathbf{x} \\ s.t. \quad & \mathbf{0} \leq \mathbf{d} \leq \mathbf{u} \\ & \mathbf{1}'\mathbf{d} \geq L \\ & x_j = \min\{d_j, y_j, c - \textstyle\sum_{i=1, i\neq j}^N x_i\} \quad j = 1, ..., N. \end{aligned} \tag{A-6}$$

Because capacity is partitioned into N buckets, i.e., $\sum_{j=1}^N y_j = c$, $x_j = \min\{d_j, y_j\}$ for all $j = 1, ..., N$, and Problem (A-6) can be simplified to:

$$\begin{aligned} \min_{\mathbf{d}} \quad & \mathbf{r}' \min\{\mathbf{d}, \mathbf{y}\} \\ s.t. \quad & \mathbf{0} \leq \mathbf{d} \leq \mathbf{u} \\ & \mathbf{1}'\mathbf{d} \geq L. \end{aligned}$$

Consider the KNAPSACK problem (see (a)) with the following QUESTION: Is there a subset $U' \subseteq U$ such that $\sum_{j \in U'} w_j \geq \kappa$ and $\sum_{j \in U'} v_j \leq b$?

Given any instance of KNAPSACK, we define the following instance of the minimum revenue RM problem (A-6): $\{1, ..., N\} = U$, $K = 1$, $L = \kappa$ and, for any $j \in U$, $u_j = w_j$, $y_j = 1$, $r_j = v_j$. Hence, the minimum revenue problem can be formulated as follows:

$$\begin{aligned} \min_{\mathbf{d}} \quad & \textstyle\sum_{j \in U} v_j \min\{d_j, 1\} \\ s.t. \quad & \mathbf{0} \leq \mathbf{d} \leq \mathbf{w} \\ & \mathbf{1}'\mathbf{d} \geq \kappa. \end{aligned} \tag{A-7}$$

Problem (A-7) is a concave minimization problem. Hence, there exists an optimal solution that is an extreme point. With a similar argument to (a), one can show that $d_j^* = 0$ or $d_j^* = w_j$ for all $j = 1, ..., N$. Let $\mathbf{d}^*$ be an optimal solution to (A-7), with optimal value greater than or equal to b. Define $U' = \{j : d_j^* = w_j\}$. Then, $\kappa \geq \sum_{j \in U} d_j^* = \sum_{j \in U'} w_j$ and $b \leq \sum_{j \in U} v_j \min\{d_j^*, 1\} = \sum_{j \in U'} v_j$; therefore, the answer to KNAPSACK is positive. Conversely, suppose the answer to KNAPSACK is positive. Setting $d_j = w_j$ for all $j \in U'$, and $d_j = 0$ for all $j \in U \setminus U'$ gives rise to a feasible solution, with an objective value greater than or equal to b. As a result, Problem (A-6) is NP-Hard.

(c) Consider the following three-product single-resource problem with partitioned booking limits (i.e., $\mathcal{S} = \{\{1\}, \{2\}, \{3\}\}$), with $\mathbf{r} = [3, 2, 1]$, $c = 6$, and $\mathcal{P} = \{\mathbf{d} | d_1 = 6, d_2 = 5, d_3 = 2\}$, that is, demand is deterministic. With $\mathbf{y}^1 = [6, 5, 1]$, $\rho(\mathbf{y}^1) = (6 \times 3) - (5 \times 2 + 1 \times 1) = 7$ and with $\mathbf{y}^2 = [6, 3, 3]$, $\rho(\mathbf{y}^2) = (6 \times 3) - (1 \times 3 + 3 \times 2 + 2 \times 1) = 7$. In contrast, with $\mathbf{y} = (\mathbf{y}^1 + \mathbf{y}^2)/2 = [6, 4, 2]$, $\rho(\mathbf{y}) = (6 \times 3) - (4 \times 2 + 2 \times 1) = 8 > \max\{\rho(\mathbf{y}^1), \rho(\mathbf{y}^2)\}$. Therefore, $\rho(\mathbf{y})$ is not quasiconvex. For the same problem instance, $\varphi(\mathbf{y}^1) = \varphi(\mathbf{y}^2) = 11 > \varphi(\mathbf{y}) = 10$. Therefore, $\varphi(\mathbf{y})$ is not quasiconcave.

□

Proof of Lemma 1.

When low-fare customers arrive first, the realized sales associated with any demand vector $\mathbf{d}$ can be recursively defined as follows:

$$x_N = d_N,$$

$$x_j \quad = \quad \min\left\{d_j, c - \sum_{k>j} x_k\right\}, \quad j = 1, ..., N-1. \tag{A-8}$$

Consider a different booking sequence for which (A-8) is not satisfied. That is, there exist some products i and j with $1 \le i < j \le N$, such that $x_j < \min\left\{d_j, c - \sum_{k>j} x_k\right\}$ if $j < N$ or $x_j < d_j$ if $j = N$ and $x_i > 0$. Swapping the arrival times of the earliest request for product i with the latest request for product j decreases the realized revenue by $r_j - r_i < 0$ without altering the perfect hindsight solution. Hence, (A-8) must be satisfied in every worst-case scenario. One can therefore assume that low-fare customers book first without loss of generality.

For any j, $1 \le j \le N$, the derivative of the minimum revenue function with respect to d_j equals

$$r_j \mathbb{1}\left\{\sum_{k\ge j} d_k \le c\right\} - \sum_{i=1}^{j-1} r_i \mathbb{1}\left\{\sum_{k>i} d_k \le c < \sum_{k\ge i} d_k\right\}.$$

For any demand vector $\mathbf{d}$, suppose that $\sum_{k\ge 1} d_k \le c$. Then, the derivative of the revenue function with respect to d_j equals $r_j > 0$ for all $j = 1, ..., N$, and the revenue is therefore minimized when $d_j = l_j$ for all $j = 1, ..., N$. Suppose now that there exists some product i, $1 \le i \le N-1$, for which $\mathbb{1}\{\sum_{k>i} d_k \le c < \sum_{k\ge i} d_k\} = 1$. The derivative of the revenue function with respect to d_j equals 0 for all $j \le i$, and $r_j - r_i < 0$ for all $j > i$. Hence, the revenue is minimized when $d_j = u_j$ for all $j = 1, ..., N$. $\square$

Proof of Lemma 2.

When low-fare customers arrive first, the realized sales associated with any demand vector $\mathbf{d}$ can be recursively defined as (A-8). By contrast, the sales under perfect hindsight are independent of the sequence of arrivals and are equal to

$$\begin{aligned} z_1 &= d_1, \\ z_j &= \min\left\{d_j, c - \sum_{k<j} z_k\right\}, \quad j = 2, ..., N. \end{aligned} \tag{A-9}$$

Using a similar interchange argument as in Lemma 2, one can show that the worst-case arrival sequence is when low-fare customers book first.

For any j, $1 \leq j \leq N$, the derivative of the regret function with respect to d_j equals:

$$\begin{aligned}
& r_j \mathbb{1}\left\{\sum_{k\leq j} d_k \leq c\right\} - \sum_{i=j+1}^{N} r_i \mathbb{1}\left\{\sum_{k<i} d_k \leq c < \sum_{k\leq i} d_k\right\} \\
& -r_j \mathbb{1}\left\{\sum_{k\geq j} d_k \leq c\right\} + \sum_{i=1}^{j-1} r_i \mathbb{1}\left\{\sum_{k>i} d_k \leq c < \sum_{k\geq i} d_k\right\} \\
\geq \quad & r_j \left(\mathbb{1}\left\{\sum_{k\leq j} d_k \leq c\right\} - \left(1 - \mathbb{1}\left\{c < \sum_{k\leq j} d_k\right\} - \mathbb{1}\left\{\sum_{k\leq N} d_k \leq c\right\}\right)\right) \\
& -r_j \left(\mathbb{1}\left\{\sum_{k\geq j} d_k \leq c\right\} - \left(1 - \mathbb{1}\left\{c < \sum_{k>j-1} d_k\right\} - \mathbb{1}\left\{\sum_{k\geq 1} d_k \leq c\right\}\right)\right) \\
= \quad & 0.
\end{aligned}$$

As a result, the maximum regret is nondecreasing with d_j for any j, $1 \leq j \leq N$, and one can therefore assume that $d_j = u_j$ for all $j = 1, ..., N$ without loss of generality. $\square$

Proof of Proposition 4.

From Lemma 2, the regret can only be reduced when x_j is smaller than (A-8) for any j, $1 \leq j \leq N$, so as to protect capacity for higher-fare products. For any set $\bar{S} \in \bar{\mathcal{S}}$, let us denote by $\bar{y}_{\bar{S}}$ the protection level for products $j \in \bar{S}$, that is, $\sum_{j\in\bar{S}} x_j \geq \bar{y}_{\bar{S}}$. Suppose that the sets $\bar{S} \in \bar{\mathcal{S}}$ are not nested. In particular, suppose that there exists a set $\bar{S} \in \bar{\mathcal{S}}$ and two products j and k with $1 \leq j < k$, such that $j \notin \bar{S}$ and $k \in \bar{S}$. For simplicity assume that $\bar{\mathcal{S}} = \{\bar{S}\}$ and $\bar{S} = \{k\}$, but the argument can easily be generalized. For any demand vector $\mathbf{d}$, the perfect hindsight revenue equals $r_j \min\{d_j, c\} + r_k \min\{d_k, c - d_j\}$ by (A-9). Under the protection level policy, the worst-case realized revenue is attained when either the total demand for product k comes first or when the total demand for product j comes first. Specifically, when $\bar{S} = \{k\}$, the realized revenue equals $r_k \min\{d_k, c\} + r_j \min\{d_j, c - d_k\}$ if product k is requested first and $r_j \min\{d_j, c - \bar{y}_{\bar{S}}\} + r_k \min\{d_k, \max\{\bar{y}_{\bar{S}}, c - d_j\}\}$ if product j is requested first. By contrast, when $\bar{S} = \{j, k\}$, the worst-case realized revenue equals $r_k \min\{d_k, c\} + r_j \min\{d_j, c - d_k\}$ and is always achieved when product k is requested first because $r_j \geq r_k$. Hence, nesting is optimal, i.e., $\bar{\mathcal{S}} = \{\{1\}, \{1, 2\}, ..., \{1, ..., N-1\}\}$.

Finally, for any $\bar{S} \in \bar{\mathcal{S}}$, the nested protection level policy can be expressed as a nested booking limit policy as follows: $\sum_{j\in S} x_j \leq y_S$ with $y_S = c - \bar{y}_{\bar{S}}$ where $S = \{1, ..., N\} \setminus \bar{S}$ (Talluri and van Ryzin 2004). $\square$

Proof of Lemma 3.

For convenience, let us denote $y_j = y_{\{j,...,N\}}$ for any j, $2 \le j \le N$. Without loss of generality, we assume that $y_j \le c$ for all $j = 2, ..., N$. Under a nested booking limit policy, the realized sales when low-fare customers book first equal

$$\begin{aligned} x_N &= \min\{d_N, y_N\} \\ x_j &= \min\left\{d_j, y_j - \sum_{k>j} x_k\right\}, \quad j = 1, ..., N-1, \end{aligned} \tag{A-10}$$

while the perfect hindsight sales equal (A-9). Similar to Lemma 1, one can show that the maximum regret is always attained when low-fare customers book first, .

Let $\{t_1, ..., t_T\}$ with $1 \le t_1 \le ... \le t_T \le N$ denote the product nest indices for which the booking limits are attained under the demand vector $\mathbf{d}$, that is, $x_{t_i} = y_{t_i} - \sum_{k>t_i} x_k$ for all $i = 1, ..., T$. Accordingly, for any demand vector $\mathbf{d}$, the regret equals

$$\begin{aligned} &\sum_{j=1}^{N} r_j \min\left\{d_j, \left(c - \sum_{k<j} d_k\right)^+\right\} - \sum_{j=1}^{t_1-1} r_j d_j \\ -\ &\sum_{i=1}^{T-1}\left(\sum_{j=t_i+1}^{t_{i+1}-1} r_j d_j + r_{t_i}\left(y_{t_i} - y_{t_{i+1}} - \sum_{j=t_i+1}^{t_{i+1}-1} d_j\right)\right) - \left(\sum_{j=t_T+1}^{N} r_j d_j + r_{t_T}\left(y_{t_T} - \sum_{j=t_T+1}^{N} d_j\right)\right). \end{aligned}$$

The derivative of the regret function with respect to d_j for any j, $1 \le j \le t_1 - 1$, equals

$$r_j \mathbb{1}\left\{\sum_{k\le j} d_k \le c\right\} - \sum_{i=j+1}^{N} r_i \mathbb{1}\left\{\sum_{k<i} d_k \le c < \sum_{k\le i} d_k\right\} - r_j \le 0,$$

while the derivative of the regret function with respect to d_j for any j, $t_i + 1 \le j \le t_{i+1} - 1$, for any i, $1 \le i \le T - 1$, and for any j, $t_T + 1 \le j \le N$, when $i = T$ equals

$$\begin{aligned} &r_j \mathbb{1}\left\{\sum_{k\le j} d_k \le c\right\} - \sum_{i=j+1}^{N} r_i \mathbb{1}\left\{\sum_{k<i} d_k \le c < \sum_{k\le i} d_k\right\} - r_j + r_{t_i} \\ \ge\ & r_j\left(\mathbb{1}\left\{\sum_{k\le j} d_k \le c\right\} - \left(1 - \mathbb{1}\left\{c < \sum_{k\le j} d_k\right\} - \mathbb{1}\left\{\sum_{k\le N} d_k \le c\right\}\right)\right) - r_j + r_{t_i} \\ \ge\ & r_j \mathbb{1}\left\{\sum_{k\le N} d_k \le c\right\} \ge 0. \end{aligned}$$

Finally, the derivative of the regret function with respect to d_j for any $j = t_i$ and for any i, $1 \le i \le T$, equals zero. Consequently, if t_1 is the largest product nest $\{t_1, ..., N\}$ for which

$x_{t_1} = y_{t_1} - \sum_{k>t_1} x_k$ in a worst-case demand scenario, then the regret is maximized when $d_j = l_j$ for all $j \le t_1 - 1$ and $d_j = u_j$ for all $j \ge t_1$.

We next show that if t_1 is the largest product nest $\{t_1, ..., N\}$ for which $x_{t_1} = y_{t_1} - \sum_{k>t_1} x_k$ for any demand vector $\mathbf{d}$, then t_1 is also the largest product nest $\{t_1, ..., N\}$ for which $x_{t_1} = y_{t_1} - \sum_{k>t_1} x_k$ when $d_j = l_j$ for all $j \le t_1 - 1$ and $d_j = u_j$ for all $j \ge t_1$. For convenience, we denote by x'_j the realized sales of product j when $d_j = l_j$ for all $j \le t_1 - 1$ and $d_j = u_j$ for all $j \ge t_1$. First, observe that for any $j \ge t_1$,

$$\begin{aligned} u_j &\ge d_j \\ &\ge y_j - \sum_{k>j} x_k = y_j - \sum_{k>j} \min\{d_k, (c - \sum_{i>k} d_i)^+\} \\ &\ge y_j - \sum_{k>j} \min\{u_k, (c - \sum_{i>k} u_i)^+\} = y_j - \sum_{k>j} x'_k; \end{aligned}$$

therefore $x'_j = y_j - \sum_{k>j} x'_k = y_j - y_{j+1}$ for all $j \ge t_1$. Second, consider the nest $\{t_1 - 1, ..., N\}$. By definition of t_1,

$$y_{t_1-1} - \sum_{k \ge t_1} x'_k = y_{t_1-1} - y_{t_1} = y_{t_1-1} - \sum_{k \ge t_1} x_k \ge d_{t_1-1} \ge l_{t_1-1};$$

hence, $x'_{t_1-1} = l_{t_1}$ if $x_{t_1-1} = d_{t_1}$. As an induction hypothesis, suppose that $x'_j = l_j$ if $x_j = d_j$ for all $j = k+1, ..., t_1 - 1$. By the induction hypothesis, we have the following:

$$y_k - \sum_{i>k} x'_i = y_k - \sum_{i=k+1}^{t_1-1} l_i - y_{t_1} \ge y_k - \sum_{i=k+1}^{t_1-1} d_i - y_{t_1} = y_k - \sum_{i>k} x_i \ge d_k \ge l_k.$$

As a result, $x'_k = l_k$ if $x_k = d_k$, completing the induction step.

Finally, the conditions for guaranteeing the existence of such a nest t_1 are (i) $u_{t_1} \ge y_{t_1} - \sum_{k>t_1} \min\{u_k, (c - \sum_{j>k} u_j)^+\}$ and (ii) $y_j - y_{j+1} \ge l_j$ for all $j = 1, ..., t_1 - 1$. Observe that (i) always holds provided $y_j \le \sum_{k \ge j} u_k$ and $y_j \le c$ for all j, which can be assumed without loss of generality. $\square$

Proof of Proposition 5.

For convenience, let us denote $y_j = y_{\{j,...,N\}}$ for any j, $2 \le j \le N$. Moreover, we define $y_1 = c$ and $y_{N+1} = 0$. According to Lemma 3, the regret is maximized when $d_j = l_j$ for all $j < t$ and $d_j = u_j$ for $j \ge t$, for any $t = 1, ..., \min\{N, \min\{k : y_k - y_{k+1} < l_k\}\}$ and the resulting sales equal $x_j = l_j$ for all $j < t$ and $x_j = y_j - y_{j+1}$ for all $j \geq t$. Accordingly, the maximum regret under the tth demand scenario, if $t \leq \min\{k : y_k - y_{k+1} < l_k\}$ can be written as follows:

$$\rho_t(\mathbf{y}) \doteq g_t - \sum_{j=t}^{N} r_j(y_j - y_{j+1}). \tag{A-11}$$

Moreover, for any $s > \min\{k : y_k - y_{k+1} < l_k\}$, the regret when $d_j = l_j$ for all $j < s$ and $d_j = u_j$ for all $j \geq s$ equals

$$\begin{aligned} \sum_{j=1}^{s-1} r_j \min\left\{l_j, \left(c - \sum_{i<j} l_i\right)^+\right\} + g_s - \sum_{j=1}^{s-1} r_j x_j - \sum_{j=s}^{N} r_j(y_j - y_{j+1}) \quad &\geq \quad g_s - \sum_{j=s}^{N} r_j(y_j - y_{j+1}) \\ &= \quad \rho_s(\mathbf{y}) \end{aligned}$$

because the right-hand side assumes that $x_j = z_j$ for all $j < s$, which is not necessarily true when $y_j - y_{j+1} < l_j$ for some products $j < s$.

On the other hand, because $s > \min\{k : y_k - y_{k+1} < l_k\}$, the regret when $d_j = l_j$ for all $j < s$ and $d_j = u_j$ for all $j \geq s$ is dominated by the regret when $d_j = l_j$ for all $j < t$ and $d_j = u_j$ for all $j \geq t$ where $t = \min\{k : y_k - y_{k+1} < l_k\}$ by Lemma 3. That is,

$$\sum_{j=1}^{s-1} r_j \min\left\{l_j, \left(c - \sum_{i<j} l_i\right)^+\right\} + g_s - \sum_{j=1}^{s-1} r_j x_j - \sum_{j=s}^{N} r_j(y_j - y_{j+1}) \quad \leq \quad \rho_t(\mathbf{y}).$$

Hence, even though $\rho_s(\mathbf{y})$ underestimates the actual regret, for any $s > \min\{k : y_k - y_{k+1} < l_k\}$, the maximum regret will never be achieved at $\rho_s(\mathbf{y})$. Hence, the maximum regret is equal to the maximum of $\rho_t(\mathbf{y})$, for $t = 1, ..., N$.

Because $\rho(\mathbf{y}) = \max_{t=1,...,N} \rho_t(\mathbf{y})$ is piecewise linear convex, Problem (4), specialized to a single-resource problem with interval uncertainty, can thus be expressed as the following LP:

$$\begin{array}{rll} \min\limits_{y_2,...,y_N} & \rho & \\ s.t. & \rho \geq g_t - \sum_{j=t}^{N} r_j(y_j - y_{j+1}) & t = 1, ..., N \\ & y_j \geq 0 & j = 2, ..., N \\ & y_1 = c, y_{N+1} = 0, & \end{array}$$

in which the first constraint ensures that $\rho(\mathbf{y}) \geq \max_{t=1,...,N} \rho_t(\mathbf{y})$. Besides the nonnegativity constraints, this problem has N inequality constraints for N variables and can therefore be solved in closed form. Its optimal solution is given in (12). $\square$

Proof of Corollary 1.

For convenience, let us denote $y_j = y_{\{j,...,N\}}$ for any j, $2 \le j \le N$. For any $t = 1, ..., N-1$, expanding g_t yields

$$\begin{aligned} g_t &= r_t \min\left\{u_t, \left(c - \sum_{i=1}^{t-1} l_i\right)^+\right\} + \sum_{j=t+1}^{N} r_j \min\left\{u_j, \left(c - \sum_{i=1}^{t-1} l_i - u_t - \sum_{i=t+1}^{j-1} u_i\right)^+\right\} \\ &\le r_t \min\left\{u_t, \left(c - \sum_{i=1}^{t-1} l_i\right)^+\right\} + \sum_{j=t+1}^{N} r_j \min\left\{u_j, \left(c - \sum_{i=1}^{t-1} l_i - l_t - \sum_{i=t+1}^{j-1} u_i\right)^+\right\} \\ &= r_t \min\left\{u_t, \left(c - \sum_{i=1}^{t-1} l_i\right)^+\right\} + g_{t+1}. \end{aligned}$$

Because, $y_t - y_{t+1} \le (g_t - g_{t+1})/r_t$, we conclude that $y_t - y_{t+1} \le \min\{u_t, (c - \sum_{i=1}^{t-1} l_i)^+\}$.

On the other hand,

$$\begin{aligned} g_t &\ge r_t \min\left\{l_t, \left(c - \sum_{i=1}^{t-1} l_i\right)^+\right\} + \sum_{j=t+1}^{N} r_j \min\left\{u_j, \left(c - \sum_{i=1}^{t-1} l_i - l_t - \sum_{i=t+1}^{j-1} u_i\right)^+\right\} \\ &= r_t \min\left\{l_t, \left(c - \sum_{i=1}^{t-1} l_i\right)^+\right\} + g_{t+1}. \end{aligned}$$

If $y_{t+1} > 0$, $y_t - y_{t+1} = (g_t - g_{t+1})/r_t \ge \min\{l_t, (c - \sum_{i=1}^{t-1} l_i)^+\}$. $\square$

Proof of Lemma 4.

For a particular demand vector $\mathbf{d}$, the perfect hindsight solution solves

$$\begin{aligned} \max_{\mathbf{z}} \quad & \textstyle\sum_{j=0}^{K} r_j z_j \\ s.t. \quad & z_j + z_0 \le c_j \quad j = 1, ..., K \\ & 0 \le z_j \le d_j \quad j = 0, ..., K. \end{aligned} \tag{A-12}$$

Let $\mathcal{T} = \{1 \le j \le K : d_j + d_0 \ge c_j\}$ denote the set of products which are capacity-constrained. Without loss of generality, we order the products such that $\mathcal{T} = \{1, ..., T\}$. We distinguish two cases, depending on whether $\sum_{j \in \mathcal{T}} r_j \ge r_0$ or not.

Case 1: $\sum_{j \in \mathcal{T}} r_j \ge r_0$**.** If $\sum_{j \in \mathcal{T}} r_j \ge r_0$, it is more profitable to allocate capacity to products in $\mathcal{T}$ rather than to the bundle. Accordingly, the realized revenue is minimized when the entire demand for the bundle comes first because the decision-maker, in the absence of control, will sell its capacity in priority to the bundle instead of protecting it to the more valuable products from $\mathcal{T}$. Accordingly, the realized sales when $\sum_{j\in\mathcal{T}} r_j \geq r_0$ are equal to:

$$\begin{aligned} x_0 &= d_0 \\ x_j &= \min\{d_j, c_j - x_0\} = \min\{d_j, c_j - d_0\} \quad j = 1, ..., K. \end{aligned} \tag{A-13}$$

Let us order products $j \in \mathcal{T}$ in increasing order of $c_j - d_j$, i.e., $c_1 - d_1 \leq c_2 - d_2 \leq ... \leq c_T - d_T$. Let

$$h = \arg\min\{1 \leq j \leq T : \sum_{i=1}^{j} r_i \geq r_0\}. \tag{A-14}$$

Clearly, $h > 1$ because $r_0 > r_j$ for any $j = 1, ..., K$. Moreover, $h \leq T$ because $\sum_{i=1}^{T} r_i \geq r_0$. Then, it is optimal to accept up to $c_h - d_h$ requests for the bundle. Because $h \in \mathcal{T}$, $c_h \leq d_h + d_0$, i.e., the demand for the bundle will always be larger than what is optimal to sell. Accordingly, the perfect hindsight solution is equal to

$$\begin{aligned} z_0 &= c_h - d_h \\ z_j &= \min\{d_j, c_j - z_0\} = c_j - c_h + d_h \quad && j = 1, ..., h-1, \\ z_j &= d_j && j = h, ..., K. \end{aligned} \tag{A-15}$$

Because $c_h - d_h \leq d_0$, $z_0 \leq x_0$ and therefore $z_j \geq x_j$ for all $j = 1, ..., K$.

Using (A-13) and (A-15), the derivative of the regret function with respect to d_0 can be expressed as

$$-\left(r_0 - \sum_{j=1}^{K} r_j \mathbb{1}\{d_j \geq c_j - d_0\}\right) = -\left(r_0 - \sum_{j\in\mathcal{T}} r_j\right) \geq 0$$

Therefore, the regret is maximized when $d_0 = u_0$.

Similarly, the derivative of the regret function with respect to d_j when $j < h$ can be expressed as

$$-r_j \mathbb{1}\{d_j \leq c_j - x_0\} = 0,$$

because $j \in \mathcal{T}$ when $j < h$. The derivative of the regret function with respect to d_j when $j > h$ can be expressed as

$$r_j \left(1 - \mathbb{1}\{d_j \leq c_j - x_0\}\right) \geq 0.$$

Finally, the derivative of the regret function with respect to d_h can be expressed as

$$r_h\left(1-\mathbb{1}\{d_h \le c_h - d_0\}\right) - \left(r_0 - \sum_{j=1}^{h-1} r_j\right) = \sum_{j=1}^{h} r_j - r_0 \ge 0,$$

in which the equality follows from the fact that $h \in \mathcal{T}$ and the inequality follows from (A-14). As a result, the regret is maximized when $d_j = u_j$ for all $j = 1, ..., K$.

Let $\mathbf{z}'$ be the perfect hindsight solution (A-15) when $\mathbf{d} = \mathbf{u}$ and let $\mathcal{T}' = \{1 \le j \le K : u_j + u_0 \ge c_j\}$. First, observe that $\mathcal{T}' \supseteq \mathcal{T}$. Therefore, $\sum_{j\in\mathcal{T}'} r_j \ge \sum_{j\in\mathcal{T}} r_j \ge r_0$, and the regret remains maximized when the total demand for the bundle arrives first.

On the other hand, the derivative of the revenue function with respect to d_0 equals

$$r_0 - \sum_{j=1}^{K} r_j \mathbb{1}\{d_j \ge c_j - d_0\} = r_0 - \sum_{j\in\mathcal{T}} r_j \le 0;$$

hence, the revenue is minimized when $d_0 = u_0$. Moreover, the derivative of the revenue function with respect to d_j, for any j, $1 \le j \le K$, equals

$$r_j \mathbb{1}\{d_j \le c_j - d_0\} \ge 0,$$

and the revenue function is minimized when $d_j = l_j$ for all $j = 1, ..., K$. This demand scenario is a worst-case demand scenario as long as $\sum_{j=1}^{K} r_j \mathbb{1}\{l_j + u_0 \ge c_j\} \ge r_0$; otherwise, the revenue could be further reduced by letting the bundle arrive last, as we analyze next.

Case 2: $\sum_{j\in\mathcal{T}} r_j < r_0$. If $\sum_{j\in\mathcal{T}} r_j < r_0$, it is more profitable to serve the demand for the bundle before the demand for products from $\mathcal{T}$. Accordingly, the realized revenue is maximized when the entire demand for the bundle comes last because the decision-maker, in the absence of control, sells its capacity to products $1, ..., K$ instead of protecting some for the more valuable product 0. Accordingly, the realized sales when $\sum_{j\in\mathcal{T}} r_j < r_0$ are equal to:

$$\begin{aligned} x_j &= d_j & j &= 1, ..., K, \\ x_0 &= \min\{d_0, \min_{k=1,...,K}\{c_k - d_k\}\}. \end{aligned} \tag{A-16}$$

By contrast, the perfect hindsight solution equals

$$\begin{aligned} z_0 &= d_0 \\ z_j &= \min\{d_j, c_j - z_0\} = \min\{d_j, c_j - d_0\} \quad j = 1, ..., K. \end{aligned} \tag{A-17}$$

Clearly, $z_0 \geq x_0$ and $z_j \leq x_j$ for all $j = 1, ..., K$.

Using (A-16) and (A-17), the derivative of the regret function with respect to d_0 can be expressed as follows

$$r_0 \left(1 - \mathbb{1}\left\{d_0 \leq \min_{k=1,\dots,K} \{c_k - d_k\}\right\}\right) - \sum_{j \in \mathcal{T}} r_j$$

and is nonnegative. Indeed, when $d_0 > \min_{k=1,\dots,K} \{c_k - d_k\}$, the derivative equals $r_0 - \sum_{j\in\mathcal{T}} r_j \geq 0$ and when $d_0 \leq \min_{k=1,\dots,K} \{c_k - d_k\}$, then $d_0 + d_k \leq c_k$ for all $k = 1, ..., K$, and $\mathcal{T} = \emptyset$. As a result, the regret is maximized when d_0 is the largest, as long as the set $\mathcal{T}$ satisfies $\sum_{j\in\mathcal{T}} r_j < r_0$. Hence,

$$d_0 = \min\left\{u_0, \arg\sup_d \left\{\sum_{j=1}^{K} r_j \mathbb{1}\{d_j + d \geq c_j\} < r_0\right\}\right\}.$$

In particular, if $d_0 < u_0$, then the regret can be further increased by modifying the sequence of arrivals, and making the demand for the bundle come first, as in Case 1.

Similarly, the derivative of the regret function with respect to d_j can be expressed as follows:

$$r_j \left(\mathbb{1}\{d_j \leq c_j - d_0\} - 1\right) + r_0 \mathbb{1}\left\{c_j - d_j \leq \min\left\{d_0, \min_{k=1,\dots,K} \{c_k - d_k\}\right\}\right\}.$$

The derivative is nonpositive whenever $x_0 < c_j - d_j$ and nonnegative otherwise. As a result, the maximum regret is maximized when $d_j = l_j$, for any j such that $c_j - d_j > x_0$, and is maximized when $d_i = u_i$ for $i = \arg\min_{k=1,\dots,K}\{c_k - d_k\}$, if $c_i - d_i \leq d_0$. Observe that the structure of the solution is preserved after the change in demand because $\min_{k=1,\dots,K;k\neq i}\{c_k - l_k\} \geq \min_{k=1,\dots,K;k\neq i}\{c_k - d_k\} \geq c_i - d_i \geq c_i - u_i$ and $u_0 \geq d_0 \geq c_i - d_i \geq c_i - l_i$.

On the other hand, the derivative of the revenue function with respect to d_0 equals

$$r_0 \mathbb{1}\left\{d_0 \leq \min_{k=1,\dots,K} \{c_k - d_k\}\right\} \geq 0$$

and the realized revenue is minimized when $d_0 = l_0$. Similarly, the derivative of the revenue function with respect to d_j is equal to:

$$r_j - r_0 \mathbb{1}\left\{c_j - d_j \leq \min\left\{d_0, \min_{k=1,\dots,K,k\neq j} \{c_k - d_k\}\right\}\right\}$$

and is positive whenever $x_0 < c_j - d_j$ and nonpositive otherwise. As a result, the realized revenue is minimized when $d_j = l_j$, for all $j = 1, ..., K$ such that $c_j - d_j > x_0$, and $d_i = u_i$ for $i = \arg\min_{k=1,\dots,K}\{c_k - d_k\}$, if $c_i - d_i \leq d_0$. $\square$

On the DAVN Booking Limits for Bundle RM

For a particular demand vector $\mathbf{d}$, the perfect hindsight solution solves (A-12). For simplicity, we assume that the optimal solution is unique and nondegenerate. Let $\mathcal{T} = \{1 \leq j \leq K : d_j + d_0 \geq c_j\}$ denote the set of products which are capacity-constrained. Without loss of generality, we order the products such that $\mathcal{T} = \{1, ..., T\}$. We distinguish two cases, depending on whether $\sum_{j \in \mathcal{T}} r_j \geq r_0$ or not.

If $\sum_{j \in \mathcal{T}} r_j \geq r_0$, the perfect hindsight solution is given by (A-15) when products $j \in \mathcal{T}$ are ordered in increasing order of $c_j - d_j$ and product h is defined by (A-14). Accordingly, the dual variables equal r_j for all all resources $j < h$, $r_0 - \sum_{j=1}^{h-1} r_j$ for resource h, and zero for all resources $j > h$. For every resource $k = 1, ..., K$ we define a modified fare for product 0, denoted by r_{0k}, by subtracting from r_0 the dual values associated with resources other than resource k (Talluri and van Ryzin 2004). Accordingly, $r_{0k} = r_k$ for all $k < h$, $r_{0h} = r_0 - \sum_{j=1}^{h-1} r_j \leq r_h$ and $r_{0k} = 0 < r_k$ for all $k > h$. For every resource $k \geq h$, the DAVN policy will therefore impose a booking limit on the number of accepted requests for product 0, i.e., $x_0 \leq y_{0k}$, unless product 0 and product k are placed in the same fare bucket.

If $\sum_{j \in \mathcal{T}} r_j < r_0$, the perfect hindsight solution is given by (A-17). Accordingly, the dual variables are equal to r_j for all $j \in \mathcal{T}$ and zero otherwise. Therefore, $r_{0k} = r_0 - \sum_{j \in \mathcal{T}: j \neq k} r_j > r_k$ for all resources $k \in \mathcal{T}$ and $r_{0k} = r_0 - \sum_{j \in \mathcal{T}} r_j > 0$ otherwise. Hence, the DAVN policy will protect some capacity for product 0 on resource k, $1 \leq k \leq K$, unless it groups product 0 and product k in the same fare buckets.

Proof of Lemma 5.

From Lemma 4, there are two worst-case demand scenarios when no control is applied.

Bundle is requested first. If the bundle is requested first, then $\sum_{j \in \mathcal{T}} r_j \geq r_0$ with $\mathcal{T} = \{1 \leq j \leq K : d_j + d_0 \geq c_j\}$. As in Lemma 4, we order products in $\mathcal{T}$ in increasing order of $c_j - d_j$ and define product h such that (A-14) holds. By (A-13), the maximum regret (resp. minimum revenue) can be reduced (resp. increased) only if the number of accepted requests for the bundle is limited by some amount, i.e., $x_0 \leq y_0$, where $y_0 \leq u_0$.

With this additional booking limit, the realized sales (A-13) are modified as follows:

$$\begin{aligned} x_0 &= \min\{d_0, y_0\} \\ x_j &= \min\{d_j, c_j - x_0\} \quad j = 1, ..., K. \end{aligned} \tag{A-18}$$

Similar to Lemma 4, one can show that the realized revenue is minimized when $d_0 = u_0$ and $d_j = l_j$ for all $j = 1, ..., N$.

From Lemma 4, we know that if $x_0 \geq z_0$, the regret is maximized when $d_j = u_j$ for $j = 0, 1, ..., K$. Hence, we assume that $x_0 \leq z_0$ in the following. Similar to Lemma 4, one can show using (A-15) and (A-18) that the derivative of the regret function with respect to d_0 equals

$$-\left(r_0 - \sum_{j \in \mathcal{T}} r_j\right) \mathbb{1}\{d_0 \leq y_0\} \geq 0.$$

Hence, the regret is maximized when $d_0 = u_0$.

Similarly, the derivative of the regret function with respect to d_j when $j < h$ equals

$$-r_j \mathbb{1}\{d_j \leq c_j - x_0\} \leq 0.$$

Hence, for any $j < h$, the regret is maximized when $d_j = \max\{l_j, c_j - c_h + d_h\}$. The derivative of the regret function with respect to d_j when $j > h$ is on the other hand equal to

$$r_j (1 - \mathbb{1}\{d_j \leq c_j - x_0\}) \leq r_j (1 - \mathbb{1}\{d_j \leq c_j - z_0\}) = 0.$$

because $x_0 \leq z_0 = c_h - d_h \leq c_j - d_j$. Finally, similar to Lemma 4, one can express the derivative of the regret function with respect to d_h as follows:

$$r_h (1 - \mathbb{1}\{d_h \leq c_h - x_0\}) - \left(r_0 - \sum_{j=1}^{h-1} r_j\right) = -\left(r_0 - \sum_{j=1}^{h-1} r_j\right) \leq 0,$$

in which the equality follows because $z_0 = c_h - d_h$ by (A-15) and $x_0 \leq z_0$ and the inequality follows from (A-14). As a result, the maximum regret is nonincreasing with d_j for any j, $1 \leq j \leq K$, and is therefore maximized when $d_j = l_j$ for all $j = 1, ..., K$. If $\sum_{j=1}^{K} r_j \mathbb{1}\{l_j + u_0 \geq c_j\} < r_0$, the regret can be further increased by modifying the sequence of arrivals, so as to make the bundle requested last, as we analyze next.

Bundle is requested last. If the bundle is requested last, then $\sum_{j \in \mathcal{T}} r_j < r_0$ with $\mathcal{T} = \{1 \leq j \leq K : d_j + d_0 \geq c_j\}$. By (A-16), the maximum regret (resp. minimum revenue) can be lowered (resp. increased) by protecting some capacity for the bundle, i.e., $x_0 \geq \bar{y}_0$, with $\bar{y}_0 \leq \min_k c_k$. With this additional control, the realized sales equal:

$$x_j = \min\{d_j, c_j - \bar{y}_0\} \qquad j = 1, ..., K, \tag{A-19}$$
$$x_0 = \min\{d_0, \min_{k=1,...,K}\{c_k - x_k\}\}.$$

Similar to Lemma 4, the revenue function is minimized when $d_0 = l_0$ and $d_j = l_j$ for all $j = 1, ..., K$ such that $x_0 > c_j - x_j$ and $d_i = u_i$ for $i = \arg\min_{k=1,...,K}\{c_k - x_k\}$, if $c_i - x_i \leq d_0$.

The derivative of the regret function with respect to d_0 equals

$$\left(r_0 - \sum_{j\in\mathcal{T}} r_j\right) - r_0 \mathbb{1}\left\{d_0 \leq \min_{k=1,...,K} c_k - x_k\right\}.$$

If $d_0 > \min_{k=1,...,K} c_k - x_k$, the derivative equals $(r_0 - \sum_{j\in\mathcal{T}} r_j) > 0$. On the other hand, when $d_0 \leq \min_{k=1,...,K} c_k - x_k$, the derivative is equal to $-\sum_{j\in\mathcal{T}} r_j < 0$. Hence, the regret is maximized at $d_0 = u_0$ when $d_0 > \min_{k=1,...,K} c_k - x_k$ and at $d_0 = l_0$ when $d_0 \leq \min_{k=1,...,K} c_k - x_k$.

On the other hand, the derivative of the regret function with respect to d_j, for any j, $1 \leq j \leq K$, equals

$$\begin{aligned} &r_j \left(\mathbb{1}\{d_j \leq c_j - d_0\} - \mathbb{1}\{d_j \leq c_j - \bar{y}_0\}\right) \\ &+ r_0 \mathbb{1}\left\{c_j - x_j \leq \min\left\{d_0, \min_{k=1,...,K}\{c_k - x_k\}\right\}\right\}\mathbb{1}\{d_j \leq c_j - \bar{y}_0\}. \end{aligned}$$

Suppose that $d_0 \leq \min_{k=1,...,K} c_k - x_k$, i.e., $x_0 = d_0$. If $d_j + \bar{y}_0 > c_j$, the derivative of the regret with respect to d_j equals $r_j\mathbb{1}\{d_j + d_0 \leq c_j\} \geq 0$. If $d_j + \bar{y}_0 \leq c_j$, i.e., $x_j = d_j$, then $d_0 \leq c_j - d_j$ because $d_0 = x_0 \leq \min_{k=1,...,K} c_k - x_k \leq c_j - d_j$, and the derivative of the regret with respect to d_j equals zero. As a result, when $d_0 \leq \min_{k=1,...,K} c_k - x_k$, the regret is maximized when $d_0 = l_0$ and $d_j = u_j$ for $j = 1, ..., K$.

Suppose that $d_0 > \min_{k=1,...,K} c_k - x_k$, i.e., $x_0 = \min_{k=1,...,K}\{c_k - x_k\} = \max\{\bar{y}_0, \min_{k=1,...,K} c_k - d_k\}$. Suppose that $x_0 = c_j - d_j \geq \bar{y}_0$ where $c_j - d_j = \min_{k=1,...,K} c_k - d_k$. Because $x_j = d_j \leq c_j - \bar{y}_0$ and $d_0 > c_j - d_j$, the derivative of the regret function with respect to d_j equals $r_0 - r_j > 0$. Hence, when $d_0 > \min_{k=1,...,K} c_k - x_k$, the regret is maximized when $x_0 = \bar{y}_0$ with $\min_{k=1,...,K} c_k - d_k < \bar{y}_0$. If $d_j + \bar{y}_0 > c_j$, the derivative of the regret with respect to d_j equals zero because $d_j + d_0 > d_j + \min_{k=1,...,K} c_k - x_k = d_j + \bar{y}_0 > c_j$. If $d_j + \bar{y}_0 < c_j$, then $x_j = d_j$ and because $x_0 = \bar{y}_0 < c_j - d_j = c_j - x_j$, the derivative of the regret function with respect to d_j equals $r_j(\mathbb{1}\{d_j + d_0 \leq c_j\} - 1) \leq 0$. Finally, when $d_j + \bar{y}_0 = c_j$, then $x_0 = c_j - d_j = c_j - x_j$, and the derivative of the regret with respect to d_j equals $r_0 - r_j > 0$. As a result, when $d_0 > \min_{k=1,...,K} c_k - x_k$, the regret is maximized when $d_0 = u_0$, $d_j = u_j$ for all $j = 1, ..., K$ such that $d_j + \bar{y}_0 \geq c_j$ with at least one i such that $d_i + \bar{y}_0 > c_i$, and $d_j = l_j$ for all $j = 1, ..., K$ such that $d_j + \bar{y}_0 < c_j$. In fact, the regret is maximized when there is only one product i such that $d_i + \bar{y}_0 > c_i$ and $d_j + \bar{y}_0 < c_j$ for all $j = 1, ..., K$, $j \neq i$

because the derivative of the regret with respect to d_j, for any $j = 1, ..., K$, is nonpositive almost everywhere, as long as $d_i + \bar{y}_0 > c_i$ for some product $i \neq j$. □

Proof of Proposition 6.

For brevity, we focus on the minimax regret. According to Lemma 5, at most $K+3$ demand scenarios must be considered to determine the optimal booking limit and protection level. Under all demand scenarios, the perfect hindsight solution solves (A-12). Let scenario $t = 0$ be the first scenario in Lemma 5, scenarios $t = 1, ..., K$ be the set of second scenarios in Lemma 5, and $t = K+1$ and $t = K+2$ be the third and fourth worst-case scenarios in Lemma 5.

Denoting by x_k^t the sales of product k under scenario t, the maximum regret equals $\rho = \max_{t=0,...,K+2}\{g^t - \sum_{j=1}^K r_j x_j^t + r_0 x_0^t\}$, in which g^t is the optimal value of (A-12) under the tth demand scenario.

In scenarios $t = 0$ and $t = K+1$, the bundle is requested first and the realized sales are given by (A-18). Under the parameter restrictions, $x_0^1 = y_0$ and $x_j^1 = \min\{l_j, c_j - y_0\}$ for all $j = 1, ..., K$ and $x_0^{K+1} = y_0$ and $x_j^{K+1} = \min\{u_j, c_j - y_0\}$ for all $j = 1, ..., K$.

In scenarios $t = 1, ..., K$ and $t = K+2$, the bundle is requested last and the realized sales are given by (A-19). Under the parameter restrictions, $x_t^t = c_t - \bar{y}_0$ and $x_j^t = l_j$ for all $j = 1, ..., K$, $j \neq t$, and $x_0^t = \bar{y}_0$, for all $t = 1, ..., K$. Note that scenarios $t = 1, ..., K$ are valid only if $u_t \geq c_t - \bar{y}_0$ and $l_j \leq c_j - \bar{y}_0$ for all $j = 1, ..., K$, $j \neq t$. Moreover, $x_j^{K+2} = \min\{u_j, c_j - \bar{y}_0\}$ and $x_0^{K+2} = l_0$. This last scenario is only valid when $\bar{y}_0 \geq l_0$.

On the real line, the quantities $c_j - l_j$ and $c_j - u_j$ for all $j = 1, ..., K$, l_0, y_0, and zero define $2K+2$ intervals. For each of these intervals, the maximum regret can be computed with an LP because the x_j^t are defined as linear functions of y_0 and $\bar{y}_0$. Given that $\bar{y}_0 \leq y_0$, there are $(2K+3)(2K+2)/2$ possible interval ranges for y_0 and $\bar{y}_0$. □

Proof of Proposition 7.

Let δ_j^0, δ_j^l, and δ_j^u the probabilities that the optimal booking limit z_j equals 0, l_j, and u_j respectively. Since any feasible value for z_j can be expressed as a convex combination of these three points, $\delta_j^0 + \delta_j^l + \delta_j^u = 1$, and these probabilities are between 0 and 1.

If $d_j \leq z_j$, the regret associated with product j equals $r_j(d_j - y_j)^+$. Maximizing this function over $l_j \leq d_j \leq z_j$ yields a maximum regret of $r_j(z_j - y_j)^+$ If $z_j \leq d_j$, the regret associated with product j equals $r_j(z_j - \min\{d_j, y_j\})$. Maximizing this function over $d_j \geq \max\{z_j, l_j\}$ yields a maximum regret of $r_j(z_j - \min\{l_j, y_j\})$ if $z_j \leq l_j$ and of $r_j(z_j - y_j)^+$ otherwise. Hence, the maximum regret is piecewise linear increasing with z_j. The maximum randomized regret therefore corresponds to the concave envelope of this function, which is also piecewise increasing with at most three breakpoints at zero, l_j, and u_j. (The regret when $z_j = y_j$ can always be replicated, or dominated, by randomizing z_j.) In particular, the regret equals $r_j(u_j - y_j)$ when $z_j = u_j$, $r_j \max\{0, l_j - y_j\}$ when $z_j = l_j$, and $-r_j \min\{y_j, l_j\}$ when $z_j = 0$. Therefore, the maximum randomized regret problem can be formulated as follows:

$$
\begin{aligned}
\max \quad & \mathbf{r}'(\mathbf{U} - \mathbf{Y})\boldsymbol{\delta}^u + \mathbf{r}' \max\{\mathbf{0}, \mathbf{L} - \mathbf{Y}\}\boldsymbol{\delta}^l - \mathbf{r}' \min\{\mathbf{L}, \mathbf{Y}\}\boldsymbol{\delta}^0, \\
s.t. \quad & \boldsymbol{\delta}^u + \boldsymbol{\delta}^l + \boldsymbol{\delta}^0 = \mathbf{1}, \\
& \mathbf{A}(\mathbf{U}\boldsymbol{\delta}^u + \mathbf{L}\boldsymbol{\delta}^l) \leq \mathbf{c}, \\
& \boldsymbol{\delta}^u, \boldsymbol{\delta}^l, \boldsymbol{\delta}^0 \geq \mathbf{0}.
\end{aligned}
$$

Let $\mathbf{p}$ and $\mathbf{q}$ be the dual variables respectively associated with the capacity constraints and the probability normalization constraints. By strong duality, the above problem is equivalent to its dual, which is a minimization problem:

$$
\begin{aligned}
\min \quad & \mathbf{p}'\mathbf{c} + \mathbf{q}'\mathbf{1} \\
s.t. \quad & \mathbf{p}'\mathbf{A}\mathbf{U} + \mathbf{q}' \geq \mathbf{r}'(\mathbf{U} - \mathbf{Y}), \\
& \mathbf{p}'\mathbf{A}\mathbf{L} + \mathbf{q}' \geq \mathbf{r}' \max\{\mathbf{0}, \mathbf{L} - \mathbf{Y}\}, \\
& \mathbf{q}' \geq -\mathbf{r}' \max\{\mathbf{Y}, \mathbf{L}\}, \\
& \mathbf{p} \geq \mathbf{0}.
\end{aligned}
$$

Plugging this inner problem into the general minimax regret problem completes the proof. $\square$